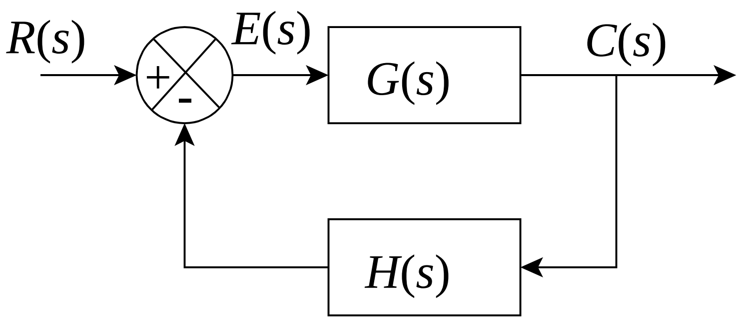
STEADY STATE ERROR ANALYSES

For the feadback system shown in block diagram below, the transfer function is given by:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$



The system error is given by:

$$\begin{aligned} E(s) &= R(s) - C(s)H(s) \\ &= \left[1 - \frac{G(s)H(s)}{1+G(s)H(s)}\right] R(s) \\ &= \frac{1}{1+G(s)H(s)} R(s) \end{aligned}$$

This last expression shows that the loop gain $G(s)H(s)$ determine the amount and nature of the steady state error of a system.

STEADY STATE ERROR ANALYSES

The loop gain $G(s)H(s)$ can be expressed in the general form;

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots\cdots(s+z_m)}{s^N(s+p_1)(s+p_2)(s+p_3)\cdots\cdots(s+p_n)}$$

$$= \frac{K\prod_{i=1}^{i=m}(s+z_i)}{s^N \prod_{j=1}^{i=n}(s+p_j)}$$

The error in this case would be given by:

$$E(s) = \frac{1}{1+G(s)H(s)} R(s)$$

$$= \frac{s^N \prod_{j=1}^{i=n}(s+p_j)}{s^N \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} R(s)$$

STEADY STATE ERROR ANALYSES

The steady state error is calculated as follows:

$$e_{ss} = \lim_{s \to 0} \left[s \frac{s^N \prod_{j=1}^{i=n} (s + p_j)}{s^N \prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} R(s) \right]$$

When the standard test signals of a step (A/s), a ramp (A/s^2), and an acceleration (A/s^3) are used, the Laplace operator "*s*" in the input test signal denominator will cancel or reduce from the power of "*s*" in the numerator of the expression above.

The power of "*s*" (the poles of the $G(s)H(s)$ located on the origin), i.e., N, determines the steady state error response of the system when subjected to standard test signals, and is called the "**type number**" of the system.

For $N = 0$, the system is a type zero, for $N = 1$, the system is a type one, and so on.

STEADY STATE ERROR ANALYSES

Type Zero System:

The steady state error for a step input; *A*/*s* is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{\prod_{j=1}^{i=n}(s+p_j)}{\prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s} \right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{\prod_{j=1}^{i=n} p_j + K\prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{1+K_p} \qquad \text{for } K_p = \frac{K\prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

STEADY STATE ERROR ANALYSES

Type Zero System:

The steady state error for a ramp input; A/s^2 is given by

$$e_{ss} = \lim_{s\to 0}\left[s \frac{\prod_{j=1}^{i=n}(s+p_j)}{\prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s^2}\right]$$

$$= \infty$$

STEADY STATE ERROR ANALYSES

Type Zero System:

The steady state error for an acceleration input; A/s^3 is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{\prod_{j=1}^{i=n} (s + p_j)}{\prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} \frac{A}{s^{3}} \right]$$

$$= \infty$$

STEADY STATE ERROR ANALYSES

Type One System:

The steady state error for a step input; *A*/*s* is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{s \prod_{j=1}^{i=n} (s + p_j)}{s \prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} \frac{A}{s} \right]$$

$$= 0$$

STEADY STATE ERROR ANALYSES

Type One System:

The steady state error for a ramp input; A/s^2 is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{s \prod_{j=1}^{i=n} (s + p_j)}{s \prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} \frac{A}{s^2} \right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{K \prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{K_v} \qquad \text{for } K_v = \frac{K \prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

STEADY STATE ERROR ANALYSES

Type One System:

The steady state error for an acceleration input; A/s^3 is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{s \prod_{j=1}^{i=n} (s + p_j)}{s \prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} \frac{A}{s^3} \right]$$

$$= \infty$$

STEADY STATE ERROR ANALYSES

Type Two System:

The steady state error for a step input; *A*/*s* is given by

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s^2 \prod_{j=1}^{i=n}(s+p_j)}{s^2 \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s}\right]$$

$$= 0$$

STEADY STATE ERROR ANALYSES

Type Two System:

The steady state error for a ramp input; A/s^2 is given by

$$e_{ss} = \lim_{s \to 0} \left[s \frac{\cancel{s^2} \prod_{j=1}^{i=n} (s + p_j)}{s^2 \prod_{j=1}^{i=n} (s + p_j) + K \prod_{i=1}^{i=m} (s + z_i)} \frac{A}{\cancel{s^2}} \right]$$

$$= 0$$

STEADY STATE ERROR ANALYSES

Type Two System:

The steady state error for an acceleration input; A/s^3 is given by

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s^2 \prod_{j=1}^{i=n}(s+p_j)}{s^2 \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s^3}\right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{K\prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{K_a} \qquad \text{for } K_a = \frac{K\prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

STEADY STATE ERROR ANALYSES

In summary;

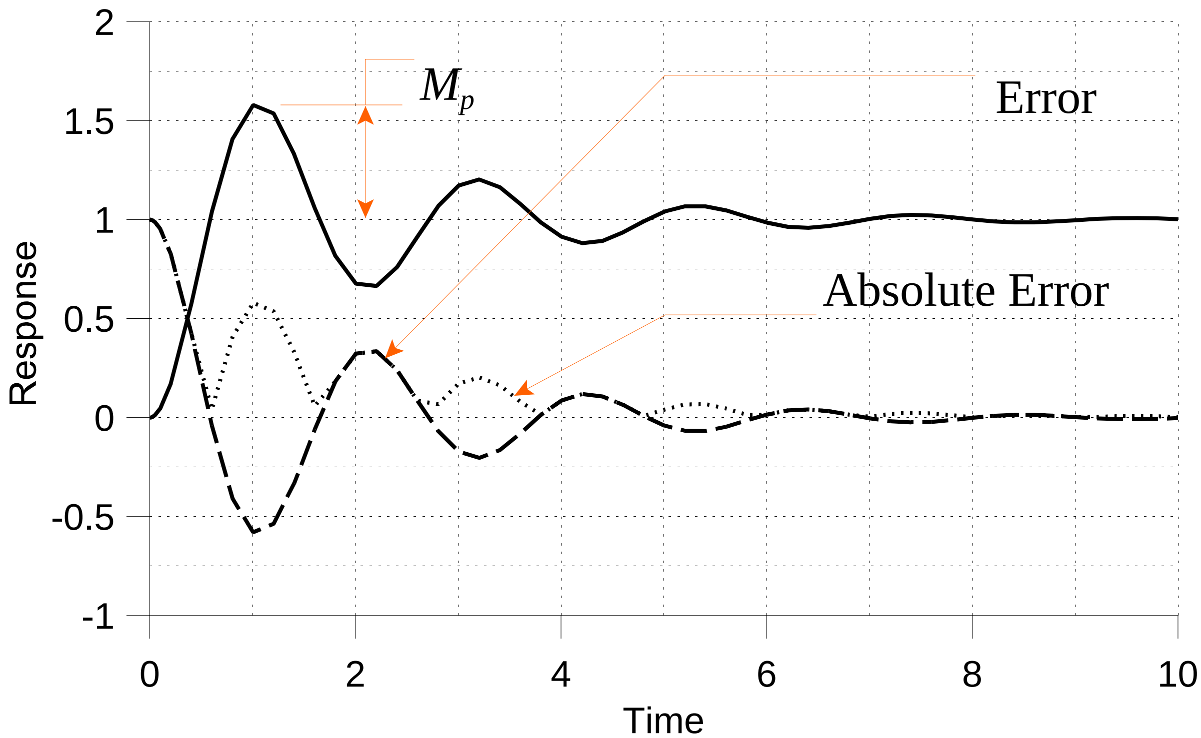
$$\text{For} \quad K_p = K_v = K_a = \frac{K \prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

TYPE	**STEP INPUT** $r(t)=A,\ R(s)=A/s$	**RAMP INPUT** $r(t)=At,\ R(s)=A/s^2$	**ACCELERATION INPUT** $r(t)=At^2,\ R(s)=A/s^3$
0	$\frac{A}{1+K_p}$	∞	∞
1	**0**	$\frac{A}{K_v}$	∞
2	**0**	**0**	$\frac{A}{K_a}$

PERFORMANCE INDICES
Error Based Criteria

Performance indices are objective quantitative measures constructed to asses the performance of systems.

System error:



PERFORMANCE INDICES
Error Based Criteria

Performance indices criterea generally used are

Maximum Deviation Error

Error Area (Integral of the deviation)

Integral Square Error.	**ASE**
Integral Absolute Error.	**IAE**
Integral Time Square Error.	**ITSE**
Integral Time Absolute Error.	**ITAE**

PERFORMANCE INDICES
Integral Square Error Based Criteria (ISE)

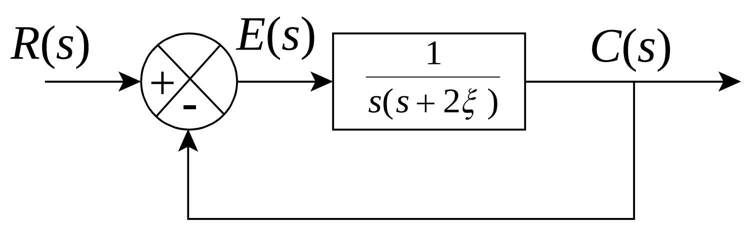
For the system shown, the error expression $E(s)$ is given by:

$$
\begin{aligned}
E(s) &= R(s) - C(s) \\
&= \left[1 - \frac{G(s)}{1+G(s)}\right] R(s) \\
&= \frac{1}{1+G(s)} R(s) \\
&= \frac{s^2 + 2\xi s}{s^2 + 2\xi s + 1} R(s)
\end{aligned}
$$

For a unit step input $R(s) = 1/s$ the error of the system is;

$$E(s) = \frac{s+\alpha}{(s+a)(s+b)}$$

where
$$
\begin{aligned}
\alpha &= 2\xi \\
a &= \xi - \sqrt{\xi^2 - 1} \\
b &= \xi + \sqrt{\xi^2 - 1}
\end{aligned}
$$



PERFORMANCE INDICES
Integral Square Error Based Criteria
(ISE)

From the inverse Laplace tables, the error in the time domain is given by:

$$e(t) = \frac{1}{b-a}\left[(\alpha - a)\, e^{-at} - (\alpha - b)\, e^{-bt}\right]$$

$$= K_1\, e^{-at} + K_2\, e^{-bt}$$

where $K_1 = \dfrac{\alpha - a}{b-a} = \dfrac{b}{b-a}$

and $K_2 = \dfrac{\alpha - b}{b-a} = -\dfrac{a}{b-a}$

The integral square error can hence be calculated as:

$$J = \int_0^\infty \left(K_1^{\,2}\, e^{-2at} + 2K_1K_2\, e^{-(a+b)t} + K_2^{\,2}\, e^{-2bt}\right) dt$$

$$= \frac{K_1^{\,2}}{2a} + \frac{2K_1K_2}{a+b} + \frac{K_2^{\,2}}{2b}$$

But $K_1K_2 = -\dfrac{ab}{4\,\xi^{\,2} - 2 - 2ab}$,

$$a+b = 2\xi$$

$$ab = 1$$

then $J = \xi + \dfrac{1}{4\xi}$

PERFORMANCE INDICES
Integral Square Error Based Criteria (ISE)

For J to be minimum, differentiate J with respect to ξ and set the result equal to zero:

$$\frac{dJ}{d\xi} = 1 - \frac{1}{4\xi^{2}} = 0$$

Solving for the damping ratio, yealds the answer that the damping ration must be 0.5 for minimum integral square error, i.e.,

$$\xi = 0.5$$

PERFORMANCE INDICES
Real Integration Theorem

If $f(t)$ is of exponential order, then the Laplace transform of the integral exists and is given by:

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

$$or \int_0^t f(t)\,dt = L^{-1}\left[\frac{F(s)}{s}\right]$$

Consequently, from the final value theorem;

$$\lim_{t\to\infty}\left[\int_0^t f(t)\,dt\right] = \lim_{s\to 0} s\,\frac{F(s)}{s}$$

$$= L^{-1}\left[\lim_{s\to 0} F(s)\right]$$

$$= \lim_{s\to 0} F(s)$$

PERFORMANCE INDICES
Laplace Approach to Calculating the Integral Square Error (ISE)

The ISE have been shown to be given by:

$$J = \int_0^\infty \left(K_1{}^2 \, e^{-2at} + 2K_1K_2 \, e^{-(a+b)t} + K_2{}^2 \, e^{-2bt} \right) dt$$

$$= \operatorname*{Lim}_{s \to 0} \left\{ L^{-1} \left(K_1{}^2 \, e^{-2at} + 2K_1K_2 \, e^{-(a+b)t} + K_2{}^2 \, e^{-2bt} \right) \right\}$$

$$= \operatorname*{Lim}_{s \to 0} \left\{ \frac{K_1{}^2}{s+2a} + \frac{2K_1K_2}{s+(a+b)} + \frac{K_2{}^2}{s+2b} \right\}$$

$$= \frac{K_1{}^2}{2a} + \frac{2K_1K_2}{a+b} + \frac{K_2{}^2}{2b}$$

But $K_1K_2 = -\dfrac{ab}{4\xi^2 - 2 - 2ab}$,

$$a+b = 2\xi$$

$$ab = 1$$

then $J = \xi + \dfrac{1}{4\xi}$

For this ISE realationship to be minimum, the damping ratio have been shown to be 0.5.

PERFORMANCE INDICES
Complex Differentation Theorem

If $f(t)$ is Laplace transformable, then except at singular points of $F(s)$;

$$\frac{dF(s)}{ds} = \frac{d\left[\int_0^\infty f(t)\, e^{-st} dt\right]}{ds}$$

$$= \int_0^\infty \frac{de^{-st}}{ds} f(t) dt$$

$$= -\int_0^\infty t\ f(t)\, e^{-st} dt$$

Consequently, the complex differentation theorem can be expressed as:

$$L\left[t\ f(t)\right] = -\frac{d\ F(s)}{ds}$$

PERFORMANCE INDICES
Laplace Approach to Calculating the Integral Time Square Error (ITSE)

For the system being analysed before, the ITSE can be expressed as:

$$
\begin{aligned}
L\left\{t[e(t)]^2\right\} &= \frac{d\,[E(s)]^2}{ds} \\
&= \frac{d}{ds}\left[\frac{K_1{}^2}{s+2a} + \frac{2K_1K_2}{s+(a+b)} + \frac{K_2{}^2}{s+2b}\right] \\
&= \frac{K_1{}^2}{(s+2a)^2} + \frac{2K_1K_2}{(s+(a+b))^2} + \frac{K_2{}^2}{(s+2b)^2}
\end{aligned}
$$

Using the real integration and the final value theorems;

$$
\begin{aligned}
\int_0^\infty t[e(t)]^2\,dt &= \operatorname*{Lim}_{s\to 0}\left[-\frac{dF(s)}{ds}\right] \\
&= \frac{K_1{}^2}{(2a)^2} + \frac{2K_1K_2}{(a+b)^2} + \frac{K_2{}^2}{(2b)^2}
\end{aligned}
$$

PERFORMANCE INDICES
Laplace Approach to Calculating the Integral Time Square Error (ITSE)

Substituting in the previous relationship for K_1, K_2, a, and b, and differentiating the result and equating it to zero gives:

$$\frac{dJ}{ds} = 0 = \frac{d}{d\xi}\left[\xi^2 + \frac{1}{8\xi^2}\right]$$

$$= 2\xi - \frac{2}{8\xi^3}$$

Solving for ξ gives its value as 0.595.

PERFORMANCE INDICES
Integral Time Absolute Error Based Criteria
(ITAE)

For the IAE the coefficients cannot be calculated analytically. The following table can be used to optimize systems based on the ITAE for a step input.

$$s + \omega_n$$

$$s^2 + 1.4\omega_n s + \omega_n^2$$

$$s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3$$

$$s^4 + 2.1\omega_n s^3 + 3.4\omega_n^2 s^2 + 2.7\omega_n^3 s + \omega_n^4$$

$$s^5 + 2.8\omega_n s^4 + 5.0\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$$

$$s^6 + 3.25\omega_n s^5 + 6.6\omega_n^2 s^4 + 8.6\omega_n^3 s^3 + 7.45\omega_n^4 s^2 + 3.95\omega_n^5 s + \omega_n^6$$

The table is utilised to match the coefficients of the denominator of the characteristic equation.

PERFORMANCE INDICES
Integral Time Absolute Error Based Criteria (ITAE)

For the ITAE the coefficients cannot be calculated analytically for optimum performance. The following table can be used to optimize systems based on the ITAE for a ramp input.

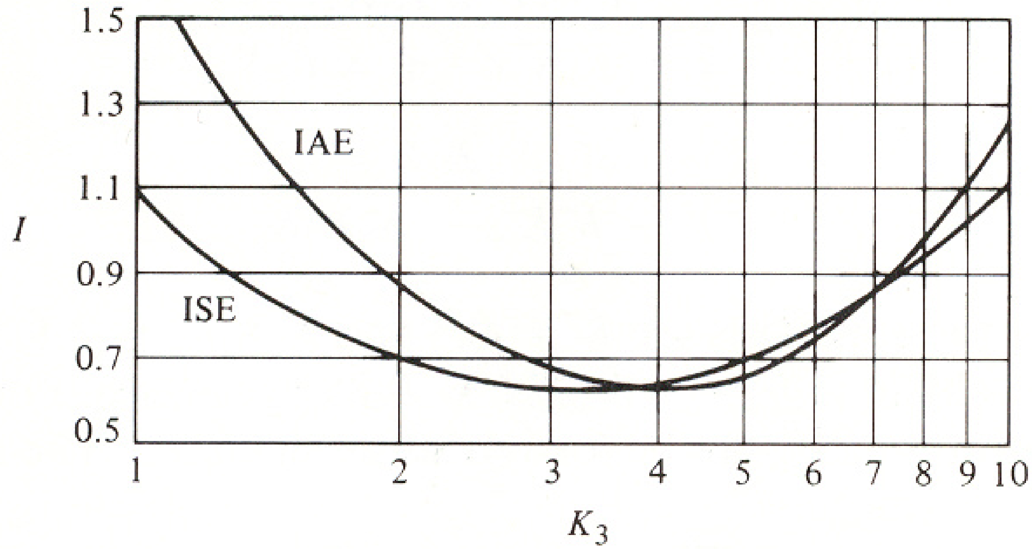
$$s^2 + 3.2\omega_n s + \omega_n{}^2$$

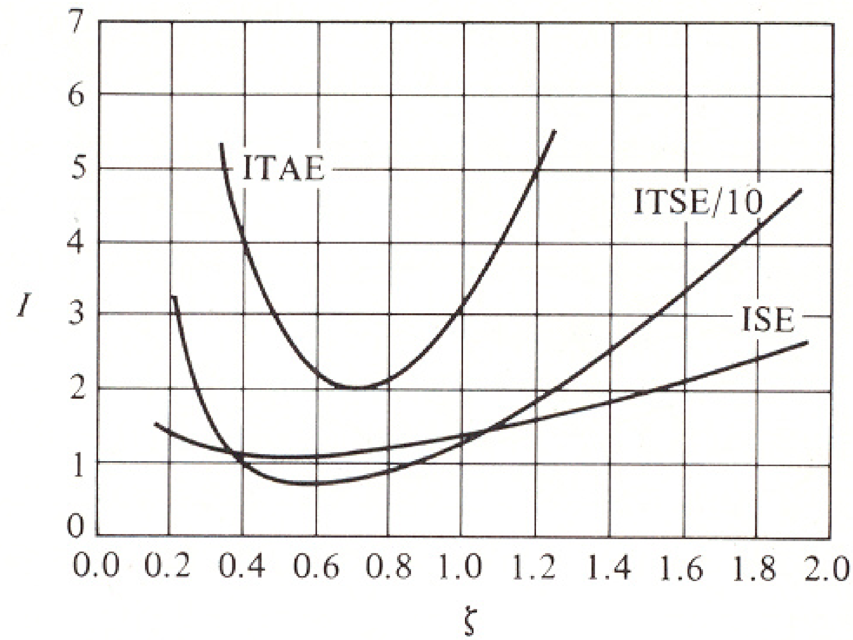
$$s^3 + 1.75\omega_n s^2 + 3.25\omega_n{}^2 s + \omega_n{}^2$$

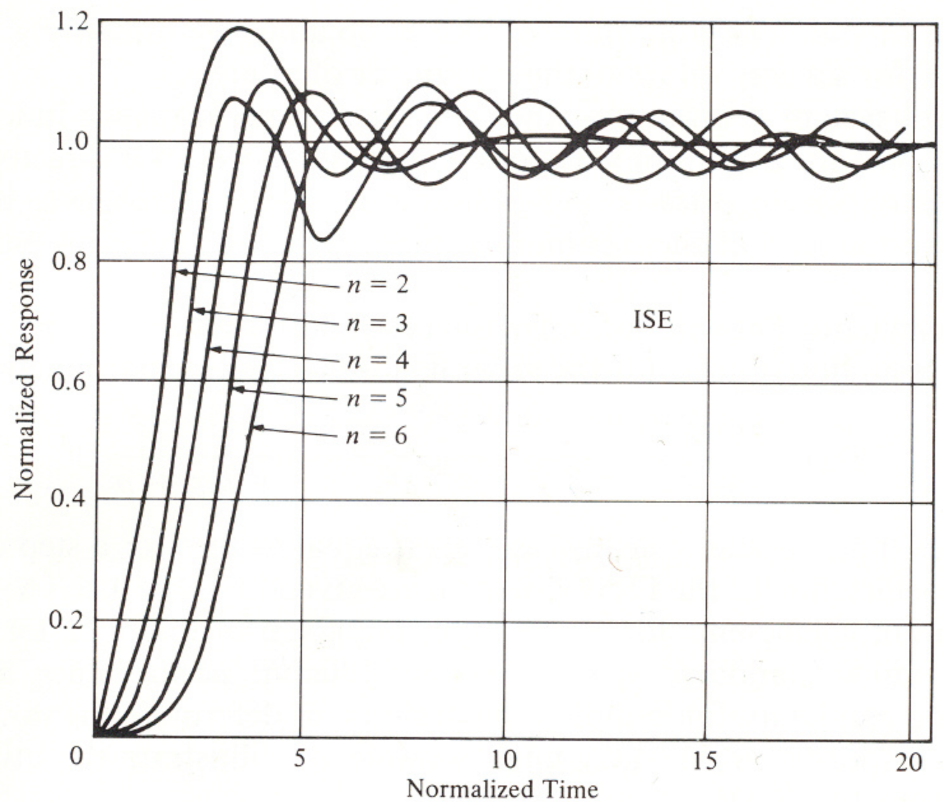
$$s^4 + 2.41\omega_n s^3 + 4.93\omega_n{}^2 s^2 + 5.14\omega_n{}^3 s + \omega_n{}^4$$

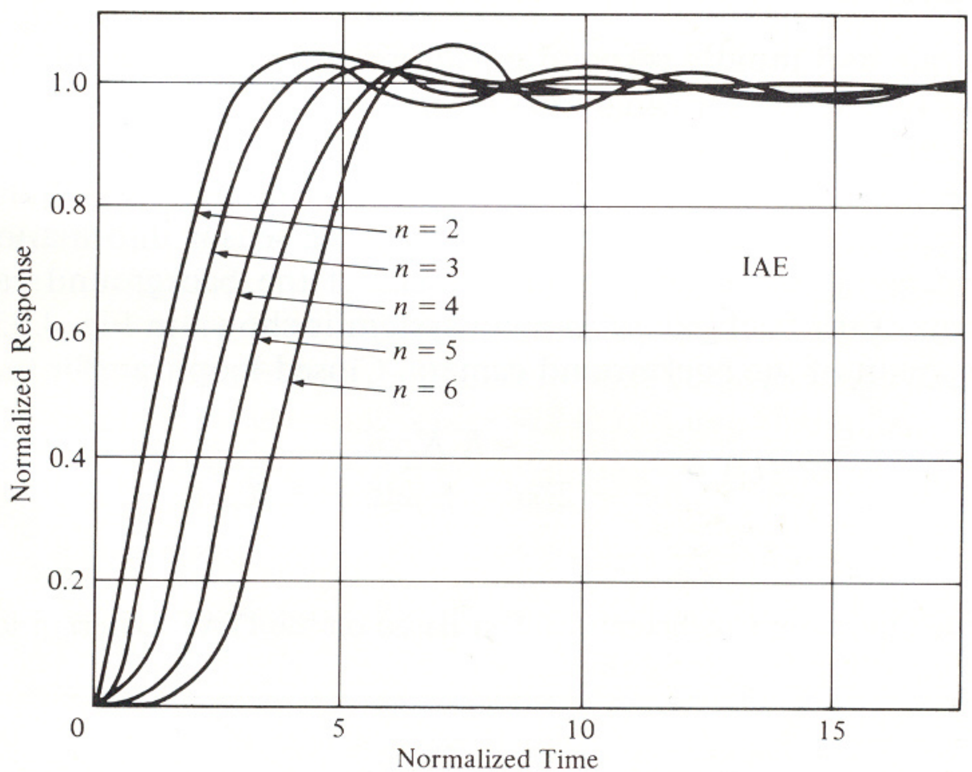
$$s^5 + 2.19\omega_n s^4 + 6.5\omega_n{}^2 s^3 + 6.3\omega_n{}^3 s^2 + 5.24\omega_n{}^4 s + \omega_n{}^5$$

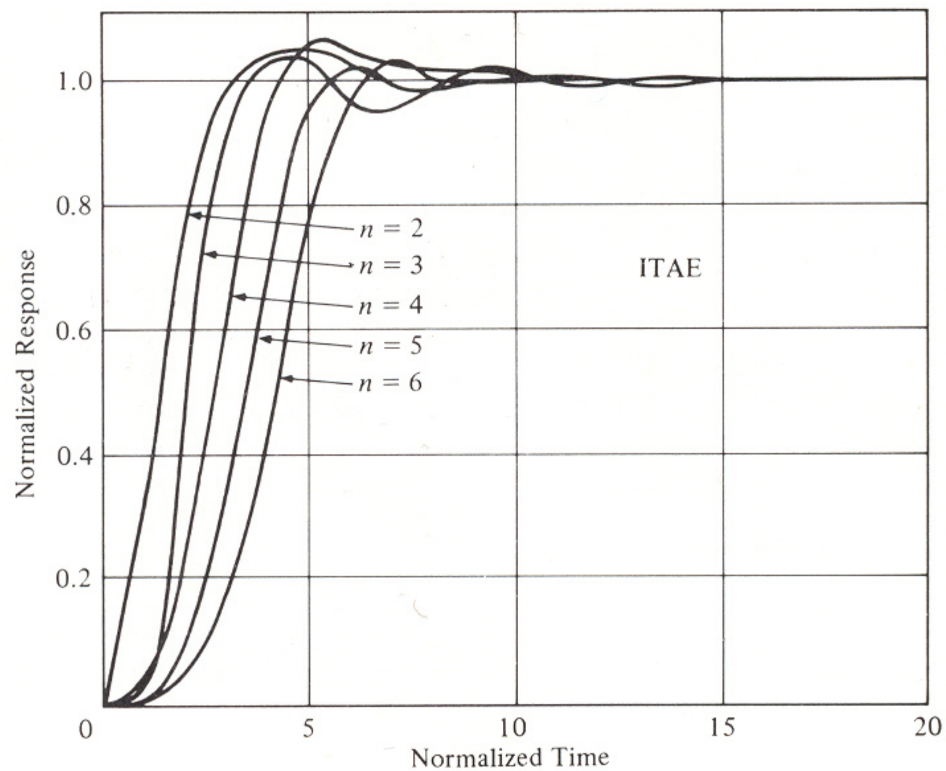
The table is utilised to match the coefficients of the denominator of the characteristic equation.


1.5
1.3
1.1
0.9
0.7
0.5
I
IAE
ISE
1
2
3
4
5
6
7
8
9
10
K_3


7
6
5
4
3
2
1
0
I
ITAE
ITSE/10
ISE
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
ζ

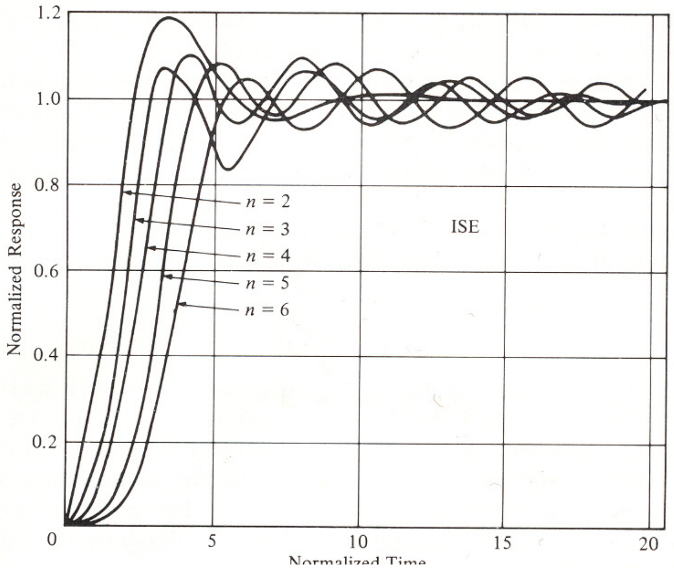

1.2
1.0
0.8
0.6
0.4
0.2
0
Normalized Response
n = 2
n = 3
n = 4
n = 5
n = 6
ISE
5
10
15
20
Normalized Time
(a)

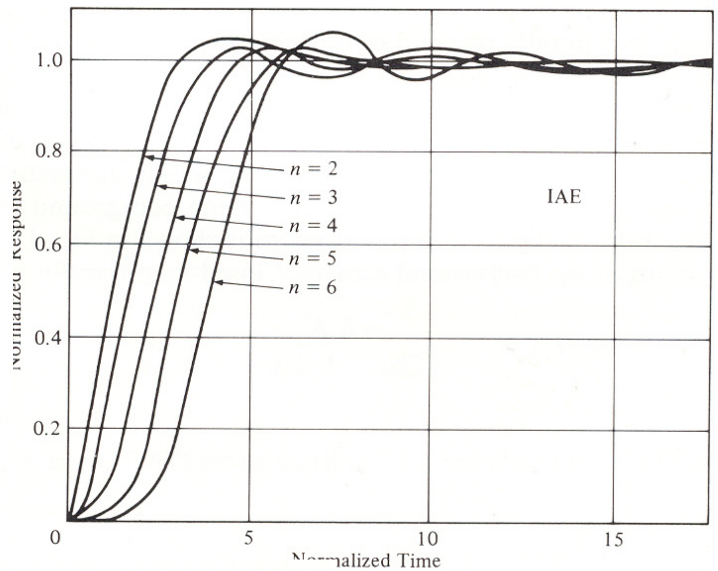

1.0
0.8
0.6
0.4
0.2
0
5
10
15
Normalized Response
Normalized Time
n = 2
n = 3
n = 4
n = 5
n = 6
IAE
(b)



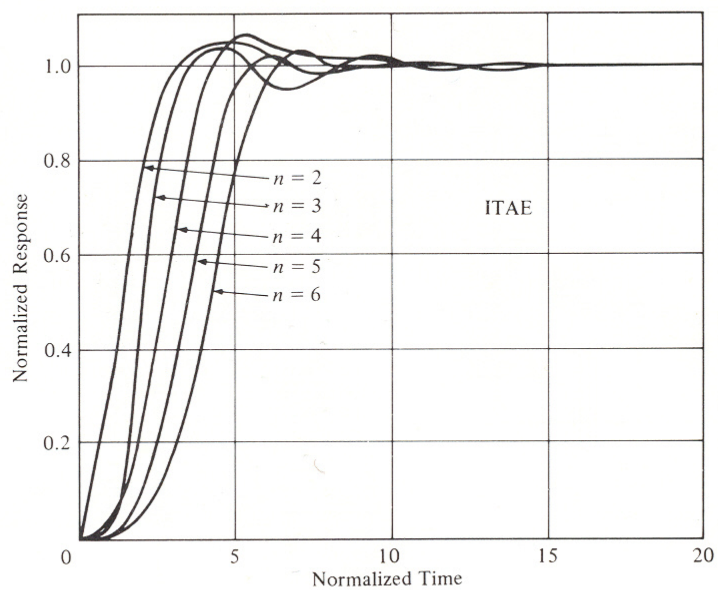
1.0
0.8
0.6
0.4
0.2
0
5
10
15
20
Normalized Response
Normalized Time
n = 2
n = 3
n = 4
n = 5
n = 6
ITAE

(c)


1.2
1.0
0.8
0.6
0.4
0.2
0
Normalized Response
n = 2
n = 3
n = 4
n = 5
n = 6
ISE
5
10
15
20


1.0
0.8
0.6
0.4
0.2
0
n = 2
n = 3
n = 4
n = 5
n = 6
IAE
5
10
15

(b)


1.0
0.8
0.6
0.4
0.2
0
Normalized Response
n = 2
n = 3
n = 4
n = 5
n = 6
ITAE
5
10
15
20
Normalized Time

(c)