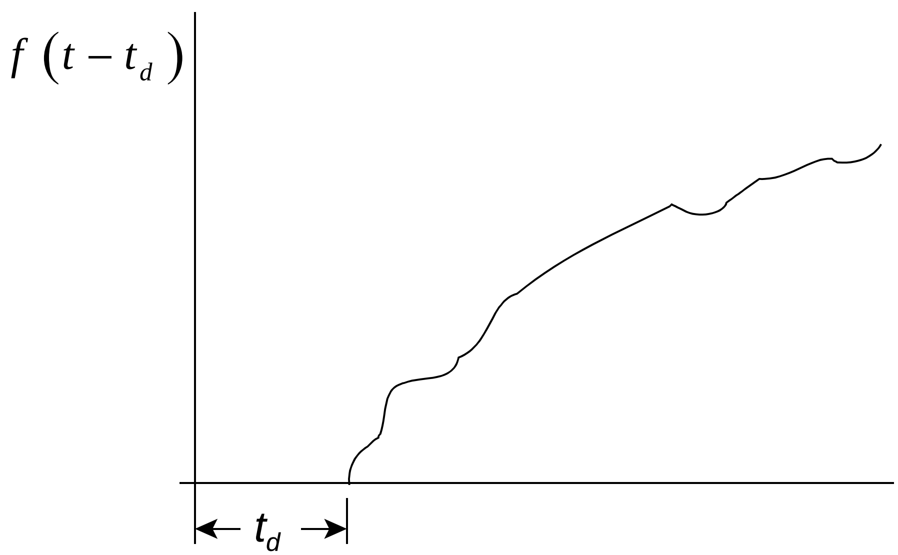
Control Systems with Transportation Lag

Systems with pure transportation lag (i.e. time delay) can be represented graphically as follows:



Control Systems with Transportation Lag

The Laplace transform of such a system is calculated from first principles as follows:

$$\mathscr{L}\left[f\left(t-t_d\right)\right] = \int_0^\infty f\left(t-t_d\right)e^{-st}dt$$

Let $\tau = t - t_d$

then $t = \tau + t_d$

and $d\tau = dt$

Substituting these quantities in the Laplace transformation equation gives:

$$\mathscr{L}\left[f\left(t-t_d\right)\right] = \int_0^\infty f\left(\tau\right)e^{-s\left(t_d+\tau\right)}d\tau$$

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The Laplace transform of the system with transportation lag is hence given by:

$$\mathscr{L}\left[f\left(t - t_d\right)\right] = \int_0^\infty f\left(\tau\right) e^{-st_d} e^{-s\tau} d\tau$$

$$= e^{-st_d} \int_0^\infty f\left(\tau\right) e^{-s\tau} d\tau$$

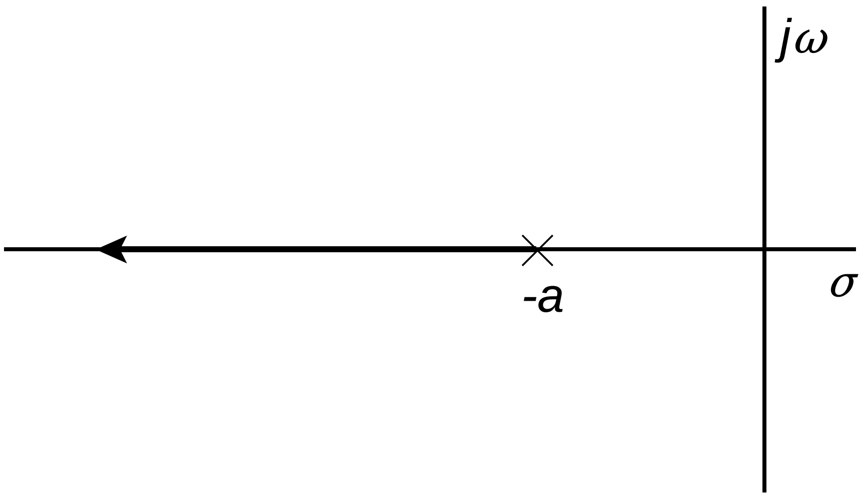
$$= e^{-st_d} F\left(s\right)$$

First Order System with Transportation Lag

Consider the following first order system:

$$\frac{C(s)}{R(s)} = \frac{K}{1+\frac{K}{(s+a)}}$$

Its root locus is as shown below;



First Order System with Transportation Lag

If the system has a transportation lag, then its characteristic equation Δ is given by:

$$\Delta = 1 + \frac{Ke^{-st_d}}{(s+a)}$$

The angle criteria in this case would be:

$$\frac{Ke^{-st_d}}{(s+a)} = -1$$

$$\left\lfloor \frac{Ke^{-st_d}}{(s+a)} \right. = \lfloor e^{-st_d} - \lfloor s+a$$

$$= \pm\pi(2k+1)$$

But $s = \sigma + j\omega$

First Order System with Transportation Lag

Substituting for *s* gives and solving:

$$\angle\frac{Ke^{-st_d}}{(s+a)} = \angle e^{-(\sigma+j\omega)t_d} - \angle s+a$$

$$= \angle e^{-\sigma t_d - j\omega t_d} - \angle s+a$$

$$= \angle e^{-j\omega t_d} - \angle s+a$$

$$= -57.7\,\omega t_d - \angle s+a$$

$$= \pm 180(2k+1)$$

Rearranging;

$$\angle s+a = \pm 180(2k+1) - 57.7\omega t_d$$

First Order System with Transportation Lag

It is important to note that the root locii for the case when $k = 0$ are known as the fundamental root locii, and are of paramount importance.

Four tasks remain to be carried out in order to draw the root locii for such a system;

1. Determine where root locii start and where do they end.
2. Calculate the break away from the real axis, if applicaable.
3. Find the intersection of the root locii with the imaginary axis, if applicable.
4. Sketch the root locii.

First Order System with Transportation Lag

When $K = 0$ the root locus starts at;

$$Ke^{-st_d} = -(s+a)$$
$$= 0$$
$$\text{i.e., } s = -a$$

But, it can also be argued that;

$$e^{-st_d} = -\left.\frac{(s+a)}{K}\right|_{k=0}$$
$$= -\infty$$

for $s = -\infty$, the limit of e^{-st_d} is given by;

$$\lim_{s \to -\infty} e^{-st_d} = \left.\frac{de^{-st_d}}{ds}\right|_{s=-\infty}$$
$$= -t_d e^{-st_d}\Big|_{s=-\infty}$$
$$= -\infty$$

First Order System with Transportation Lag

When $K = \infty$ the root locus ends at;

$$e^{-st_d} = -\frac{(s+a)}{K}$$
$$= 0$$
$$\text{i.e., } s = \infty$$

Thus a root locus start from the pole at -*a* and ends at a zero at $s = \infty$. Another root locus starts from a pole at $s = -\infty$ and ends at a second zero at $s = -\infty$.

First Order System with Transportation Lag

The breakaway point from the real axis can be calcualted as follows:

$$K = -\frac{(s+a)}{e^{-st_d}}$$

$$\therefore \frac{dK}{ds} = -\frac{\left[e^{-st_d} \times 1 - (s+a)\left(-t_d e^{-st_d}\right)\right]}{e^{-2st_d}}$$

$$= -\frac{\left[1 + st_d + at_d\right]}{e^{-st_d}}$$

$$= 0$$

$$\therefore \left[1 + st_d + at_d\right] = 0$$

On the real axis $s = \sigma_b$, thus;

$$\sigma_b = -\frac{\left[1 + at_d\right]}{t_d}$$

First Order System with Transportation Lag

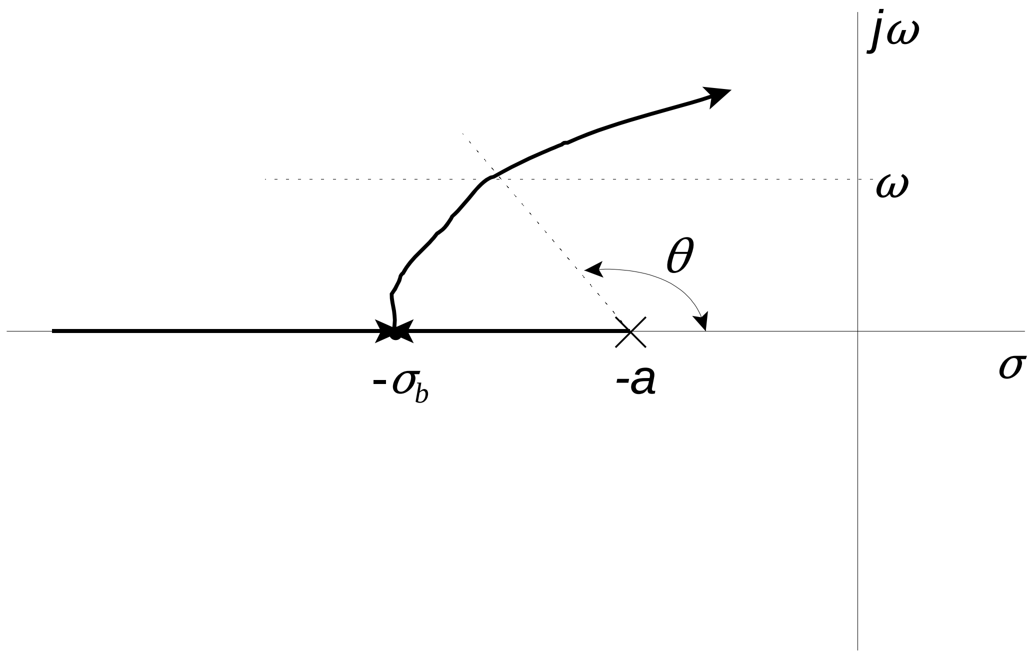
The angle from the real pole to points on the root locus for different values of *k* tabulated below:

Angle	$t_d = 1$		$t_d = 2$	
	k=0	k=1	k=0	k=1
0.2	168.46		156.92	
0.4	156.92		133.84	
0.6	145.38		110.76	
0.8	133.84		87.68	
1	122.3		64.6	
1.2	110.76		41.52	
1.4	99.22		18.44	
1.6	87.68			
1.8	76.14			
2	64.6			
2.2	53.06			
2.4	41.52			
2.6	29.98			
2.8	18.44			
3	6.9	366.9		
6		193.8		
6.2		182.26		
6.4		170.72		
6.6		159.18		
6.8		147.64		
7		136.1		
7.2		124.56		
7.4		113.02		
7.6		101.48		
7.8		89.94		
8		78.4		
8.2		66.86		
8.4		55.32		
8.6		43.78		
8.8		32.24		
9		20.7		

First Order System with Transportation Lag

The root loci are drawn as follows:

1. From the preceding table draw a horizontal line from the $j\omega$ axis at ω.
2. From the pole on the real axis $-a$ draw the angle corresponding to ω from the preceding table.
3. A point on the root locus exist at the intersection of the two lines.
4. Continue along the same procedure and join all the points.



First Order System with Transportation Lag

Giuded by the results from the preceding procedure, the value of ω at which the root locus intersect the imaginary axis can be itterated for.

$$\underline{|s+a} = \pm 180(2k+1) - 57.7\omega t_d$$

On the imaginary axis $s = \omega$, and;

$$\begin{aligned} \underline{|s+a} &= \underline{|\omega + a} \\ &= \tan^{-1}\frac{\omega}{a} \\ &= 180 - 57.7\omega t_d \end{aligned}$$

First Order System with Transportation Lag

Example of the root locus of a first order system with transportation lag.

$$\frac{C(s)}{R(s)} = \left. \frac{Ke^{-st_d}}{1+\frac{K}{(s+1)}} \right|_{t_d=1}$$