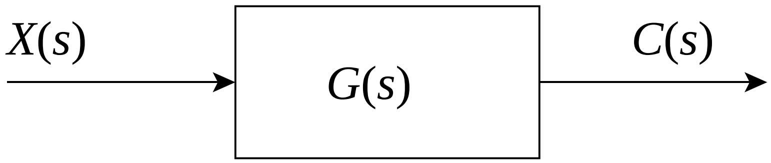
FREQUENCY RESPONSE ANALYSIS
Introduction

In this type of analysis, the steady state response of the system to a constant frequency input is investigated.

That is to say the system is subjected to the sinusoidal input, and the response of the system after the transients have passed is portrayed graphically and analysed.

To understand this approach, consider the system shown below



FREQUENCY RESPONSE ANALYSIS
Introduction

The transfer function of this system can be written in the general form as:

$$G(s) = \frac{p(s)}{q(s)}$$

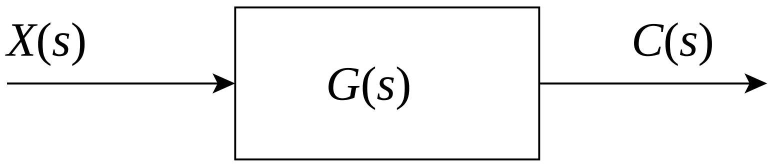
$$= \frac{p(s)}{(s+s_1)(s+s_2)\cdots\cdots(s+s_n)}$$

Assume that the system is subjected to a sinusoidal input given by:

$$x(t) = X \sin \omega t$$

i.e.,

$$X(s) = \frac{\omega X}{s^2+\omega^2}$$



FREQUENCY RESPONSE ANALYSIS
Introduction

The output of the system would hence be given by:

$$Y(s) = \frac{p(s)}{q(s)} \times X(s)$$

$$= \frac{p(s)\omega X}{(s+j\omega)(s-j\omega)(s+s_1)(s+s_2)\cdots\cdots(s+s_n)}$$

Using partial fractions, $Y(s)$ can be written as:

$$Y(s) = \frac{a}{(s+j\omega)} + \frac{\bar{a}}{(s-j\omega)} + \frac{b_1}{(s+s_1)} + \frac{b_2}{(s+s_2)} + \cdots\cdots + \frac{b_n}{(s+s_n)}$$

FREQUENCY RESPONSE ANALYSIS
Introduction

Using Laplace transforms table, $y(t)$ can be obtained from the previous expression as:

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{-s_1 t} + b_2 e^{-s_2 t} + \cdots\cdots + b_n e^{-s_n t}$$

The terms $b_1 e^{-s_1 t},\ b_2 e^{-s_2 t},\ \cdots\cdots,\ b_n e^{-s_n t}$ vanish as t tends to infinity, and $y(t)$ will tend to:

$$y(t)\Big|_{t\to\infty} = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

The constants a and $\bar{a}$ are evaluated from the partial fractions as follows:

$$a = G(s)\,\frac{\omega X}{s^2+\omega^2}\,(s+j\omega)\Big|_{s=-j\omega}$$

$$= -\frac{XG(-j\omega)}{2j}$$

$$\bar{a} = \text{conjugate of } a = \frac{XG(j\omega)}{2j}$$

FREQUENCY RESPONSE ANALYSIS
Introduction

$G(j\omega)$ and its conjugate $G(-j\omega)$ in the previous expression are complex vaiables and can be written as:

$$
\begin{aligned}
G(j\omega) &= |G(j\omega)| \times e^{j\phi} \\
&= |G(j\omega)| \times \angle G(j\omega) \\
G(-j\omega) &= |G(-j\omega)| \times e^{-j\phi} \\
&= |G(-j\omega)| \times \angle G(-j\omega)
\end{aligned}
$$

Substituting these expressions into the expressions of a and $\bar{a}$, and substituting these into the expression for $y(t)$ gives:

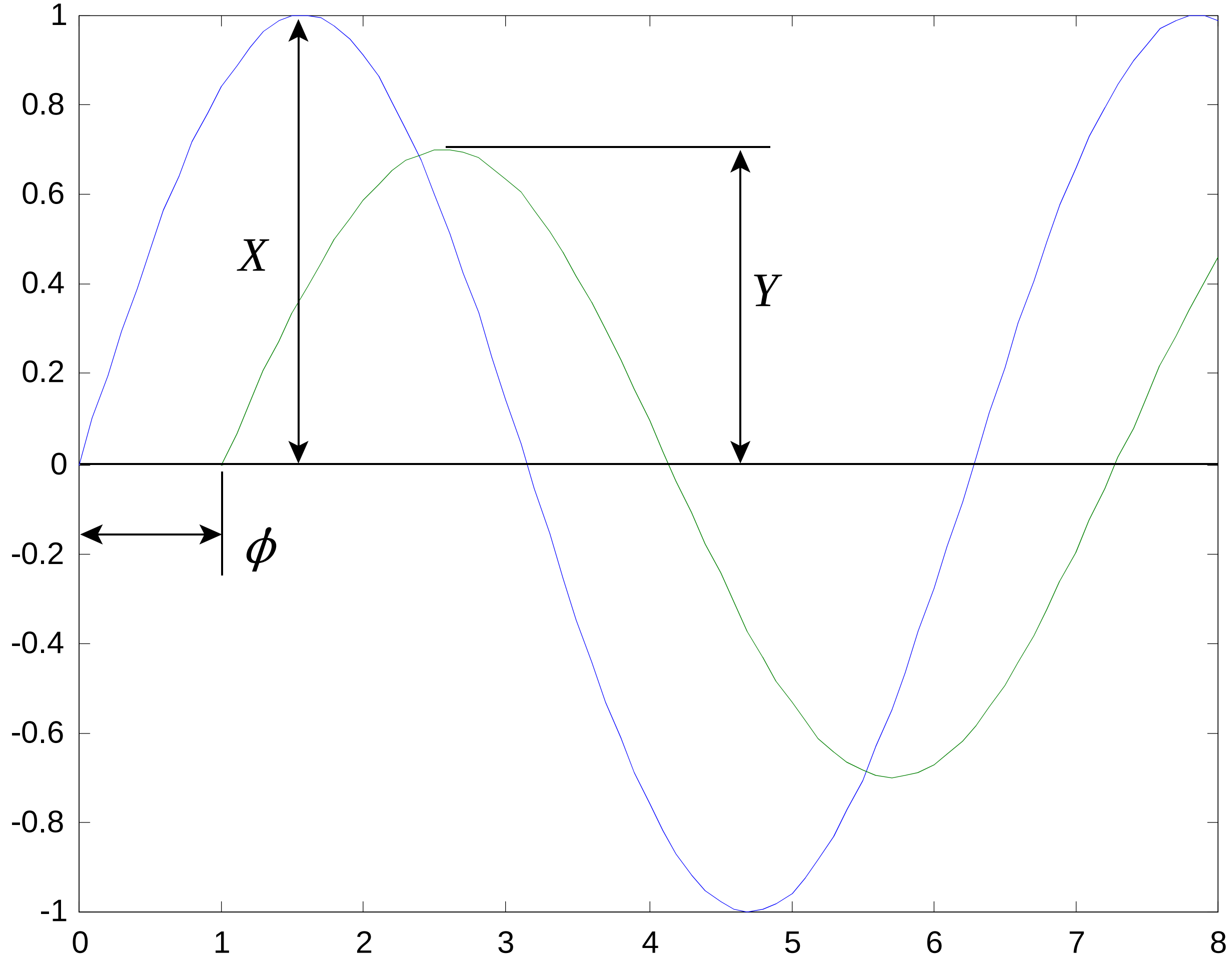
$$
\begin{aligned}
y(t) &= X|G(j\omega)| \frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\
&= X|G(j\omega)| \sin(\omega t + \phi) \\
&= Y \sin(\omega t + \phi)
\end{aligned}
$$

FREQUENCY RESPONSE ANALYSIS
Introduction

The system transfer function can hence be written as a magnitude and an angle expressed as:

$$\begin{aligned}
|G(j\omega)| &= \left|\frac{Y(j\omega)}{X(j\omega)}\right| \\
&= \frac{\text{output amplitude}}{\text{input amplitude}} \\
\angle G(j\omega) &= \angle \frac{Y(j\omega)}{X(j\omega)} \\
&= \phi \\
&= \text{phase shift between the output and the input}
\end{aligned}$$

FREQUENCY RESPONSE ANALYSIS
Introduction



FREQUENCY RESPONSE ANALYSIS

For the analysis of frequency response of control systems, three types of plots are commonly used. These are:

- Polar plots
- Bode plots
- Logmagnitude-phase plots

All these plots contain the same amount and type of information, namely:
The magnitude versus the input frequency.
The phase shift versus the frequency.

Any of these plots can be drawn from any other one, the easiest to draw and the most commonly used however is the Bode plot.

FREQUENCY RESPONSE ANALYSIS

Polar Plots:

Assume a linear time invarient simple system whose transfer function is given by:

$$\frac{Y(s)}{X(s)} = G(s)$$

The transfer function is in the form of a complex variable and is composed of a real part and an imaginary part as follows:

$$\begin{aligned} G(s)\big|_{s=j\omega} &= G(j\omega) \\ &= R(j\omega) + jx(j\omega) \\ &= \mathrm{Re} + \mathrm{Im} \end{aligned}$$

FREQUENCY RESPONSE ANALYSIS
Polar Plots

When the system is subjected to a sinusoidal input given by:

$$x(t) = X \sin \omega t$$

Will result in an output given by:

$$y(t) = Y \sin(\omega t + \phi)$$

Where:

$$Y = X\left|G(j\omega)\right| = X\sqrt{(\mathrm{Re})^2 + (\mathrm{Im})^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

FREQUENCY RESPONSE ANALYSIS
Polar Plots

Polar plots are drawn by plotting the magnitude and angle of the system transfer function for different frequencies and then joining these points. As an example, we will draw the polar plot of the following system transfer function:

$$G(s)\big|_{s=j\omega} = \left.\frac{k}{s(s+a)}\right|_{s=j\omega}$$

$$= \frac{K}{j\omega\ (j\omega\tau + 1)}$$

where

$$\tau = \frac{1}{a}$$

$$K = \frac{k}{a}$$

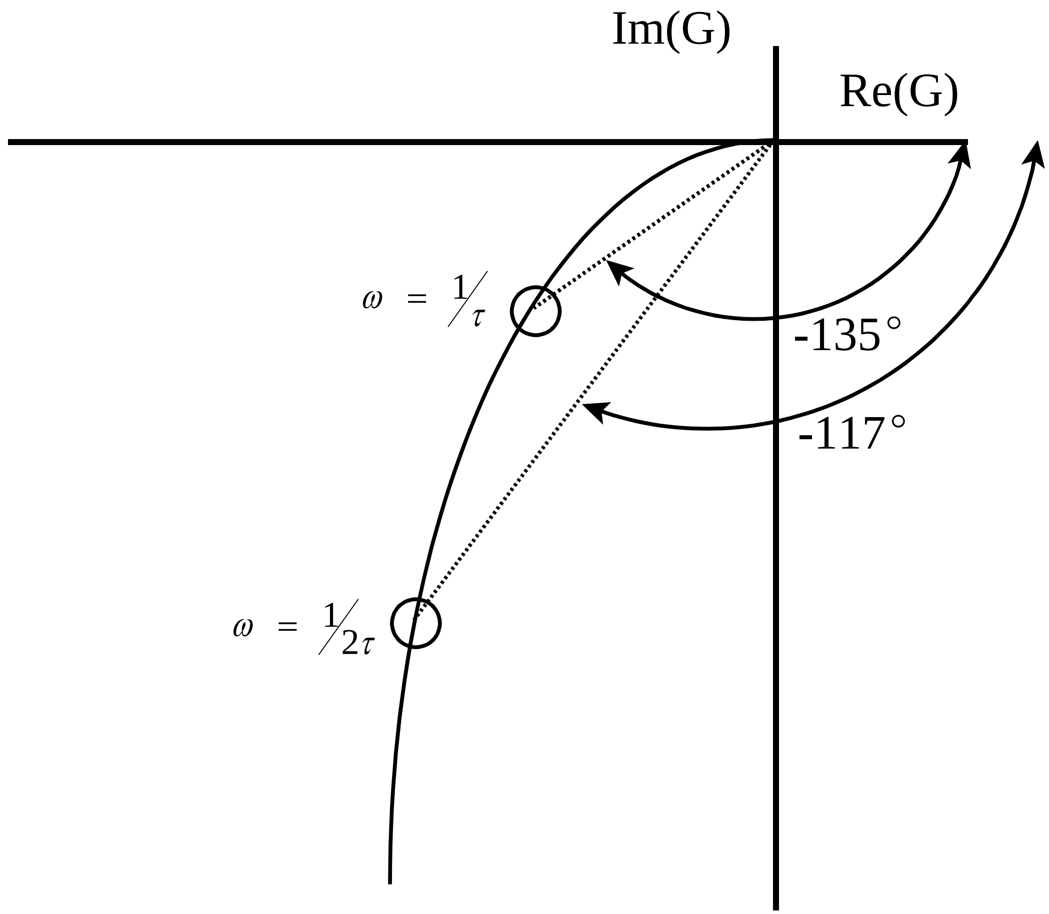
FREQUENCY RESPONSE ANALYSIS
Polar Plots

The transfer function can be written as:

$$
\begin{aligned}
G(j\omega) &= \frac{K}{-\omega^2\tau + j\omega} \\
&= \left|G(j\omega)\right| \angle G(j\omega) \\
&= \left|\frac{K}{\sqrt{\omega^2 + \omega^4\tau^2}}\right| \times -\tan^{-1}\left[\frac{1}{-\omega\tau}\right]
\end{aligned}
$$

ω	0	$1/2\tau$	$1/\tau$	∞
$\left\|G(j\omega)\right\|$	∞	$4K\tau/\sqrt{5}$	$K\tau/\sqrt{5}$	0
$\angle G(j\omega)$	- 90°	-117°	-135°	-180°

FREQUENCY RESPONSE ANALYSIS
Polar Plots



FREQUENCY RESPONSE ANALYSIS
Bode Plots

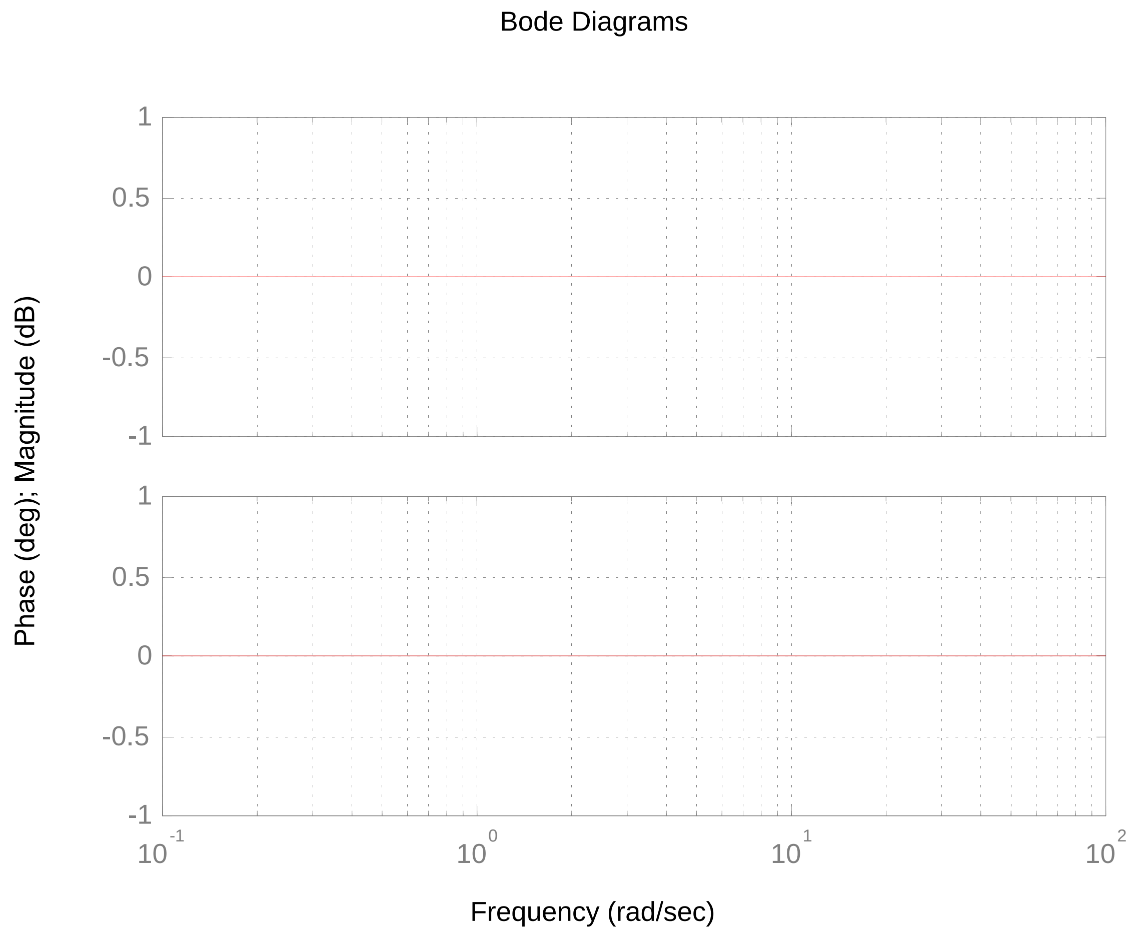
During his analysis of control systems transfer functions, Bode realized that the function can be cast in the form of:

$$G(j\omega) = \frac{K\prod_{i=1}^{m_1}[j\omega]\prod_{i=1}^{m_2}[1+j\omega\tau_i]\prod_{i=1}^{m_3}\left[1+\left(2\xi_i/\omega_{n_i}\right)j\omega+\left(j\omega/\omega_{n_i}\right)\right]}{\prod_{j=1}^{n_1}[j\omega]\prod_{j=1}^{n_2}\left[1+j\omega\tau_j\right]\prod_{j=1}^{n_3}\left[1+\left(2\xi_j/\omega_{n_j}\right)j\omega+\left(j\omega/\omega_{n_j}\right)\right]}$$

He proposed that logarithms can be used to transform the operations of multiplications and divisions into summations and subtractions.

He also proposed that graphical methods can be used to carry out the operations by plotting two graphs.

FREQUENCY RESPONSE ANALYSIS
Bode Plots



In the first, 20×logarithm of the magnitude of the response (20 log $|G(j\omega)|$), a quantity called decibels (db), is plotted versus ω on a semi-log graph paper.

In the second, the angle of the response is plotted versus ω on a semi-log graph paper.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

The plots of the different elements of the transfer function are carried out as follows:

Constant Gain:

The gain is a rational number, and as such it has a magnitude but no angle, i.e.,

Magnitude $20 \times \log K$ db

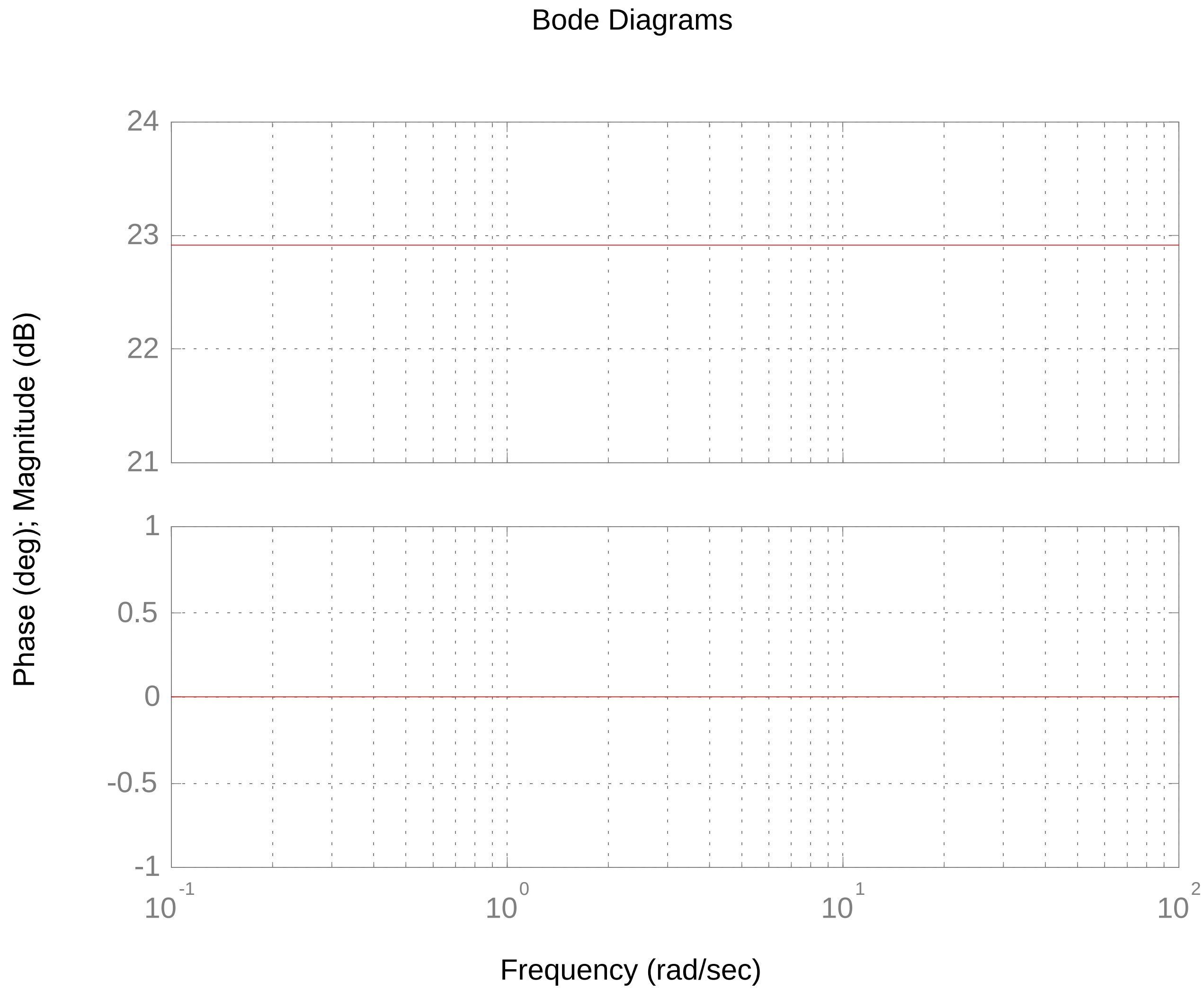
Phase Shift $0°$

This indicates that the gain of the system will cause a constant bias from zero in the magnitude curve but no change in the phase.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Constant Gain:

For k = 14



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles at the Origin:

Magnitude $\quad 20 \times \log\left|\frac{1}{[j\omega]^N}\right| = -20 \times N \times \log[\omega]$

at $\omega = 1 \quad -20 \times N \times \log[\omega] = 0$

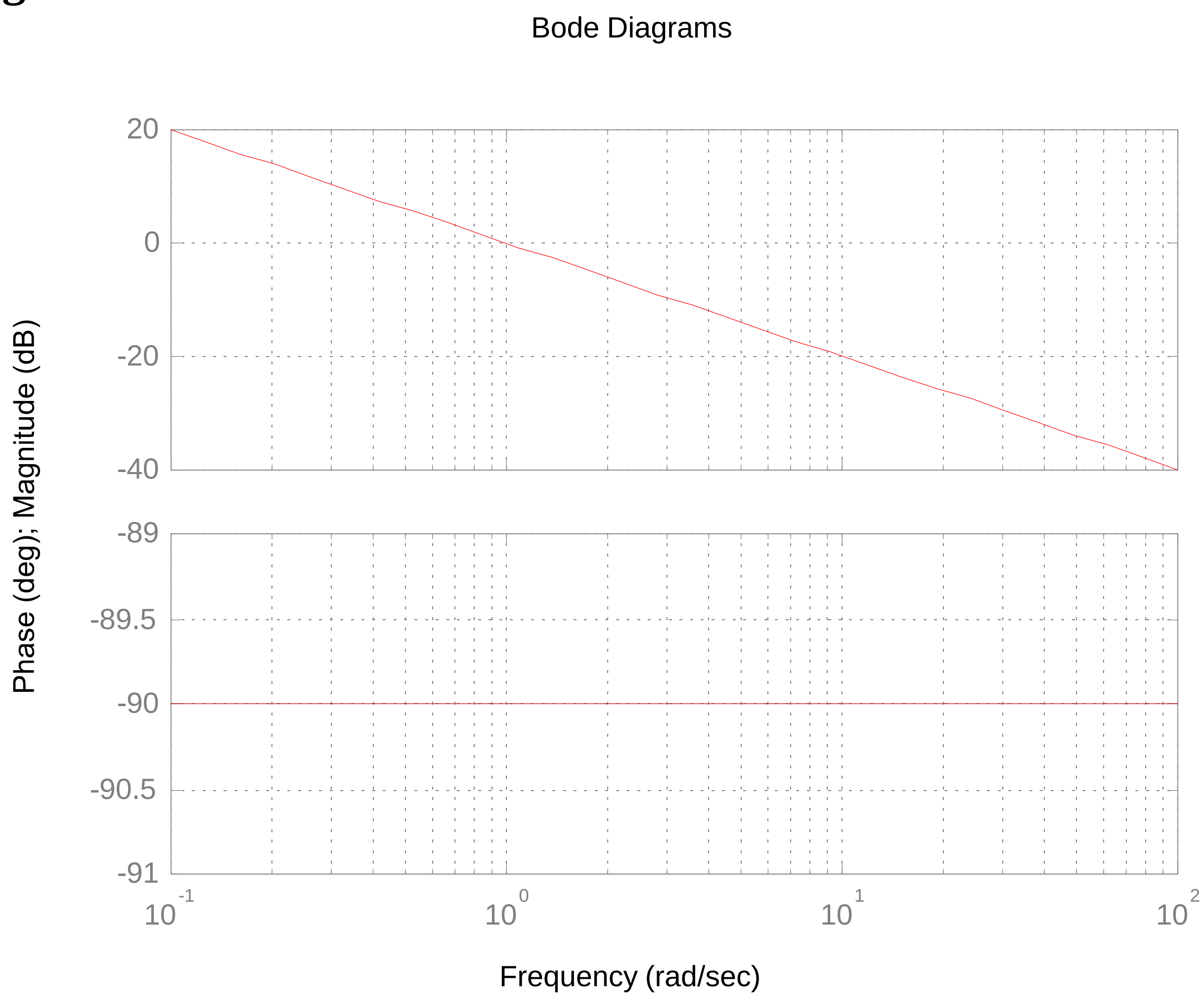
at $\omega = 10 \quad -20 \times N \times \log[\omega] = -20N$ db

Phase Shift $\quad -N \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = -N \tan^{-1}\left(\frac{\omega}{0}\right)$

$$= -90° \times N$$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles at the Origin:



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles at the Origin:

Poles at the origin will result in a decline in magnitude of 20db per decade (i.e., -20 db/ten fold increase in frequency). The magnitude is zero db at $\omega = 1$rad/sec.

The pole will also result in a constant phase shift of $-90°$.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros at the Origin:

Magnitude $\quad 20 \times \log\left|(j\omega)^N\right| = 20 \times N \times \log[\omega]$

at $\omega = 1 \quad 20 \times N \times \log[\omega] = 0$

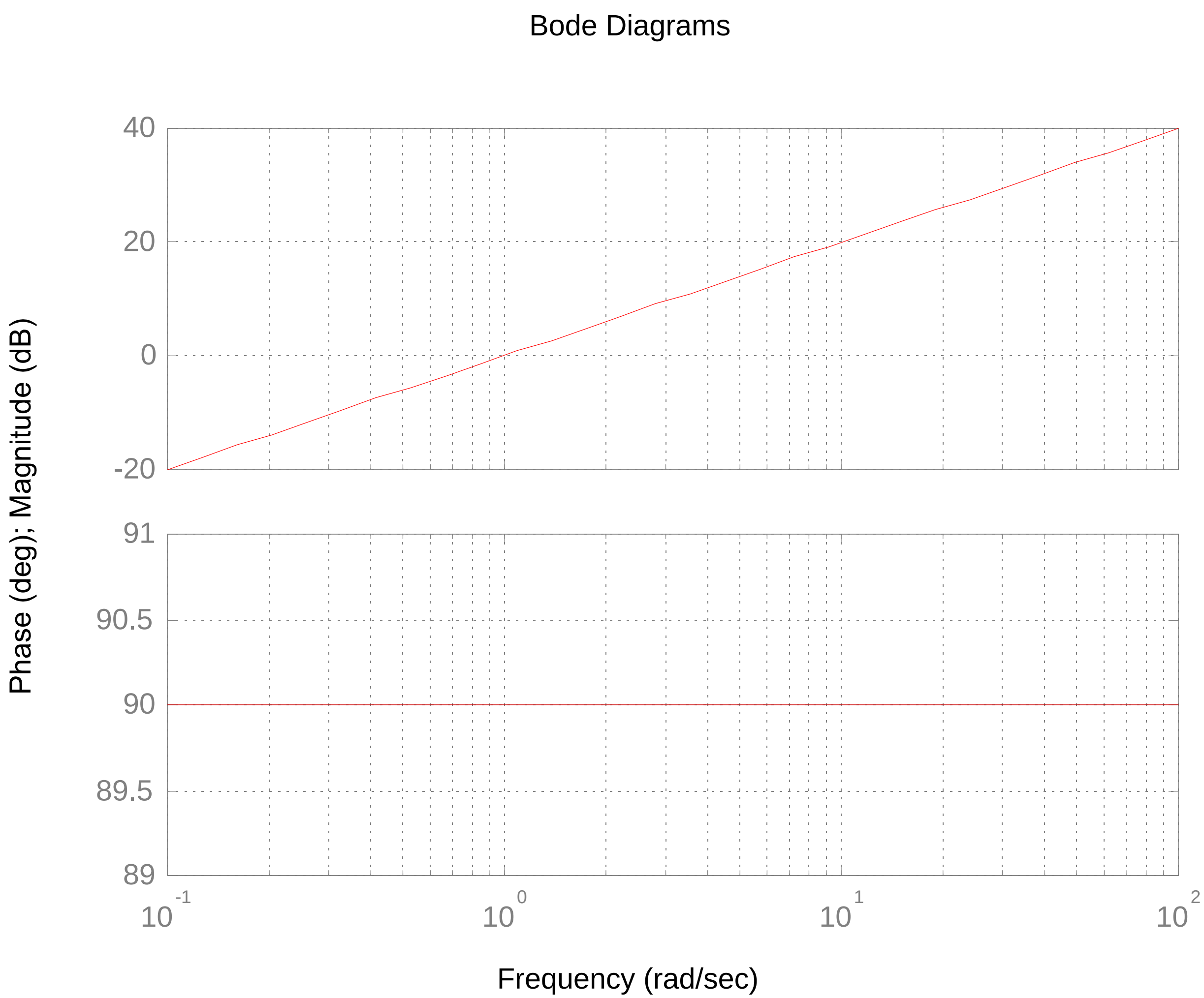
at $\omega = 10 \quad 20 \times N \times \log[\omega] = 20N$ db

Phase Shift $\quad N \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = N \tan^{-1}\left(\frac{\omega}{0}\right)$

$$= 90° \times N$$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros at the Origin:



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros at the Origin:

Poles at the origin will result in an increase in magnitude of 20db per decade (i.e., 20 db/ten fold increase in frequency). The magnitude is zero db at $\omega = 1$rad/sec.

The pole will also result in a constant phase shift of $90°$.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles on the Real Axis:

Magnitude $$20 \times \log\left|\frac{1}{(1+j\omega\tau)}\right| = 20 \times \log\left|\frac{(1-j\omega\tau)}{(1+j\omega\tau)(1-j\omega\tau)}\right|$$

$$= 20 \times \log\left|\frac{(1-j\omega\tau)}{1+\omega^2\tau^2}\right| = 20 \times \log\sqrt{\frac{1+\omega^2\tau^2}{(1+\omega^2\tau^2)^2}}$$

$$= -10 \times \log(1+\omega^2\tau^2)$$

at $\omega << \frac{1}{\tau}$ $\quad -10 \times \log(1+\omega^2\tau^2) = 0 \text{ db}$

at $\omega = \frac{1}{\tau}$ $\quad -10 \times \log(1+\omega^2\tau^2) = -3 \text{ db}$

at $\omega >> \frac{1}{\tau}$ $\quad -10 \times \log(1+\omega^2\tau^2) = -20 \times \log(\omega\tau)$

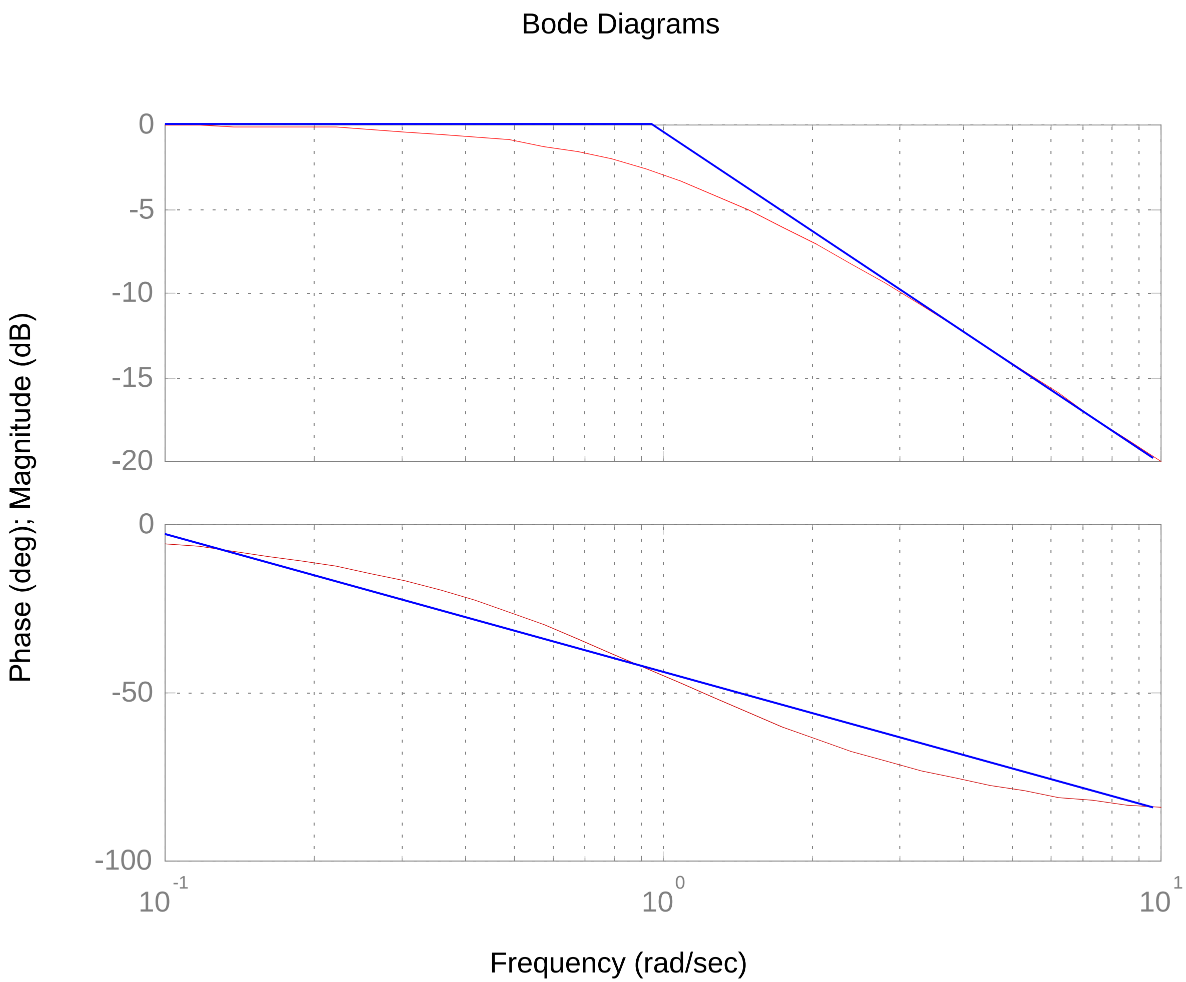
$$= -20 \text{ db for } \omega = 10/\tau$$

$$= -40 \text{ db for } \omega = 100/\tau$$

Phase Shift $\phi(\omega) = -\tan^{-1}(\omega\tau)$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles on the Real Axis:



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Poles on the Real Axis:

A poles on the real axis will result in an approximate decrease in magnitude of 20db per decade (i.e., -20 db/ten fold increase in frequency) starting after $\omega = 1/\tau$ rad/sec.

The magnitude is approximately 0 db till $\omega = 1/\tau$ rad/sec.

The maximum error between the asymptotic approximation and the exact curve is -3 db and occur at $\omega = 1/\tau$ rad/sec

The pole will also result in a phase shift of an approximate linear -90° from $\omega = 0.1/\tau$ to $\omega = 10/\tau$.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros on the Real Axis:

Magnitude $20 \times \log\left|(1 + j\omega\tau)\right| = 20 \times \log\left|\sqrt{1+\omega^2\tau^2}\right|$

$= 10 \times \log(1+\omega^2\tau^2)$

at $\omega << \dfrac{1}{\tau}$ $\quad 10 \times \log(1+\omega^2\tau^2) = 0$ db

at $\omega = \dfrac{1}{\tau}$ $\quad 10 \times \log(1+\omega^2\tau^2) = 3$ db

at $\omega >> \dfrac{1}{\tau}$ $\quad 10 \times \log(1+\omega^2\tau^2) = 20 \times \log(\omega\tau)$

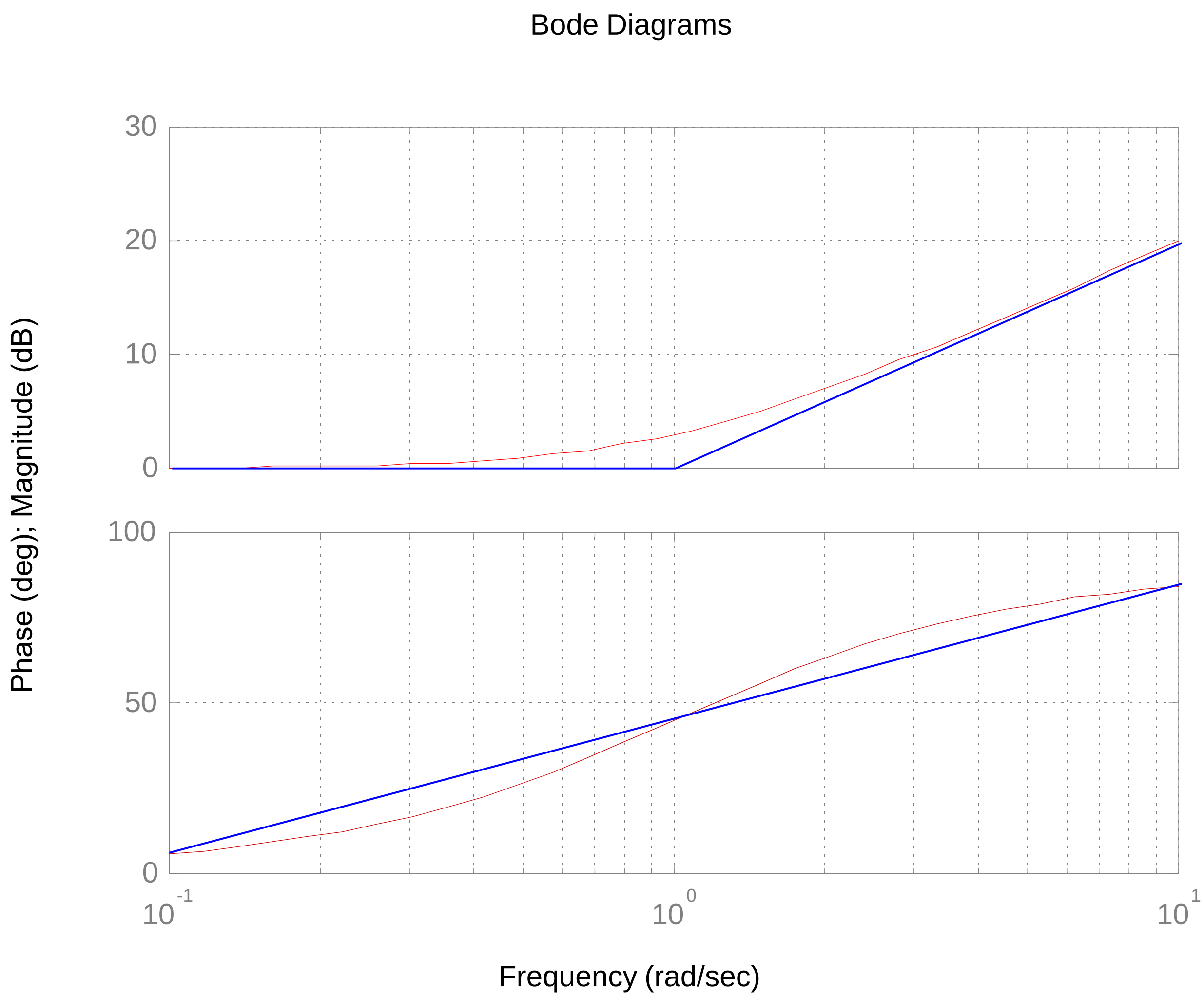
$= 20$ db for $\omega = 10/\tau$

$= 40$ db for $\omega = 100/\tau$

Phase Shift $\phi(\omega) = \tan^{-1}(\omega\tau)$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros on the Real Axis:



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Zeros on the Real Axis:

A zero on the real axis will result in an approximate increase in magnitude of 20db per decade (i.e., 20 db/ten fold increase in frequency) starting after $\omega = 1/\tau$ rad/sec.

The magnitude is approximately 0 db till $\omega = 1/\tau$ rad/sec.

The maximum error between the asymptotic approximation and the exact curve is 3 db and occur at $\omega = 1/\tau$ rad/sec

The pole will also result in a phase shift of an approximate linear 90° from $\omega = 0.1/\tau$ to $\omega = 10/\tau$.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Poles:

Magnitude $$20\times\log\left|\frac{1}{1+2\zeta\ uj-u^2}\right| = 20\times\log\left|\frac{1}{(1-u^2)+2\zeta\ uj}\right|$$

$$=20\times\log\left|\frac{(1-u^2)-2\zeta\ uj}{\big((1-u^2)+2\zeta\ uj\big)\big((1-u^2)-2\zeta\ uj\big)}\right| = 20\times\log\sqrt{\frac{(1-u^2)^2+(2\zeta\ u)^2}{\left[(1-u^2)^2+(2\zeta\ u)^2\right]^2}}$$

$$=-10\times\log\left[(1-u^2)^2+(2\zeta\ u)^2\right]$$

at $u \ll 1$ $$-10\times\log\left[(1-u^2)^2+(2\zeta\ u)^2\right] = 0\ \text{db}$$

at $u = 1$ $$-10\times\log\left[(1-u^2)^2+(2\zeta\ u)^2\right] = -10\times\log(4\zeta^2)$$

at $u \gg 1$ $$-10\times\log\left[(1-u^2)^2+(2\zeta\ u)^2\right] = -40\times\log u$$

$$=-40\ \text{db for } \omega = 10$$

$$=-80\ \text{db for } \omega = 100$$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Poles:

Phase Shift $\phi\ (j\omega)\ =\ -\tan^{-1}\left(\frac{2\zeta\ u}{1-u^2}\right)$

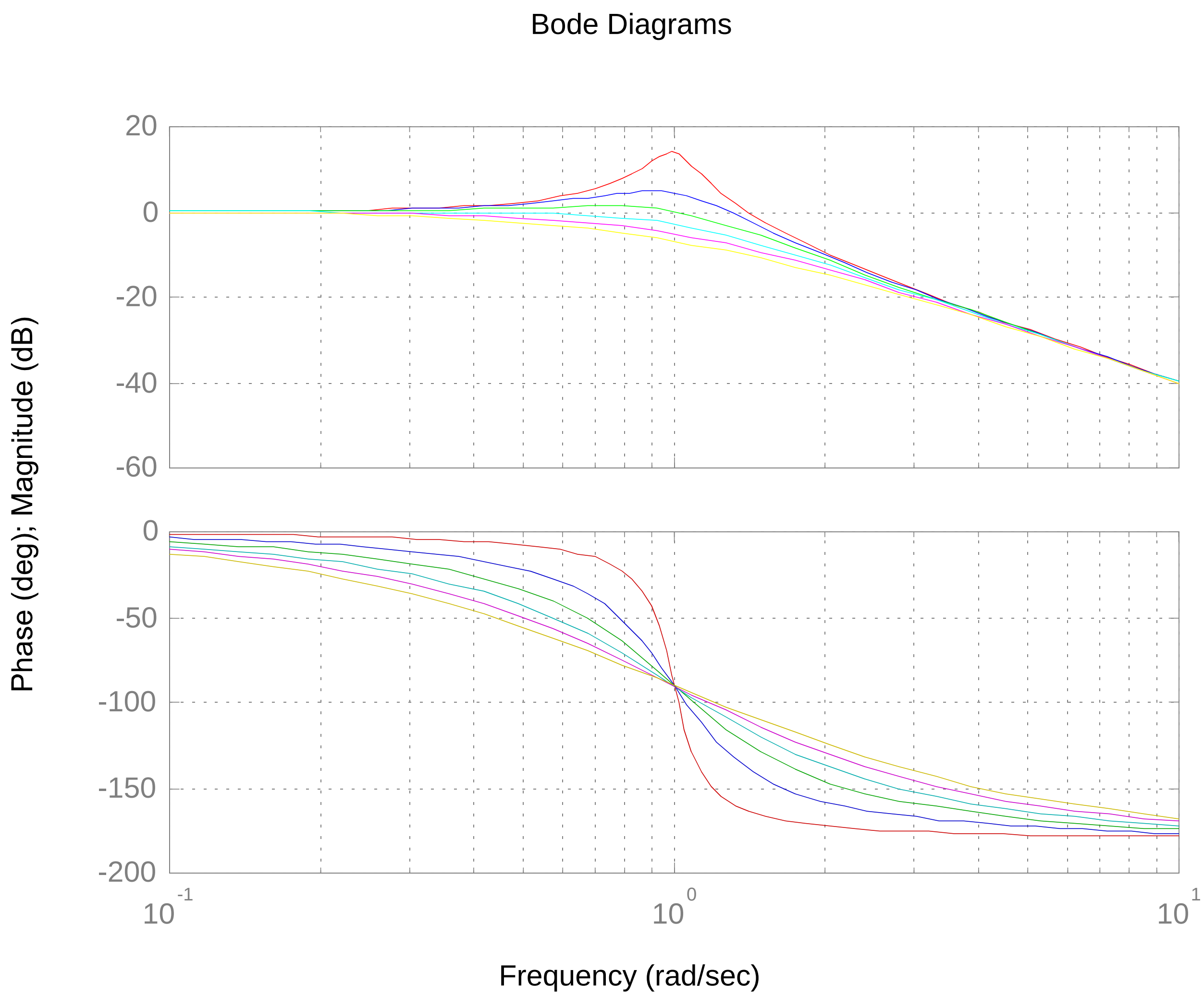
for $u\ <<\ 1\quad \phi\ (j\omega)\ =\ 0°$

for $u\ >>\ 1\quad \phi\ (j\omega)\ =\ -180°$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Poles:

ζ = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1



FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Poles:

Complex conjugate poles with a damping factor more than 0.7 will result in an approximate decrease in magnitude of -20db per decade (i.e., 20 db/ten fold increase in frequency) starting after u = 1 rad/sec.

The magnitude is approximately 0 db till u = 1 rad/sec.

The magnitude plot for complex conjugate poles with a damping factor less than 0.7 is a strong function of the damping ratio.

The poles phase shift is a strong function of the damping factor.

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Zeros:

Magnitude $$20 \times \log\left|1 + 2\zeta\, uj - u^2\right| = 20 \times \log\left|(1-u^2) + 2\zeta\, uj\right|$$

$$= 20 \times \log\sqrt{(1-u^2)^2 + (2\zeta\, u)^2} = 10 \times \log\left[(1-u^2)^2 + (2\zeta\, u)^2\right]$$

at $u \ll 1$ $$10 \times \log\left[(1-u^2)^2 + (2\zeta\, u)^2\right] = 0 \text{ db}$$

at $u = 1$ $$10 \times \log\left[(1-u^2)^2 + (2\zeta\, u)^2\right] = 10 \times \log(4\zeta^2)$$

at $u \gg 1$ $$10 \times \log\left[(1-u^2)^2 + (2\zeta\, u)^2\right] = 40 \times \log u$$

$$= 40 \text{ db for } \omega = 10$$

$$= 80 \text{ db for } \omega = 100$$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

Complex Conjugate Zeros:

$$\text{Phase Shift} \quad \phi\ (j\omega)\ =\ \tan^{-1}\left(\frac{2\zeta\ u}{1-u^2}\right)$$

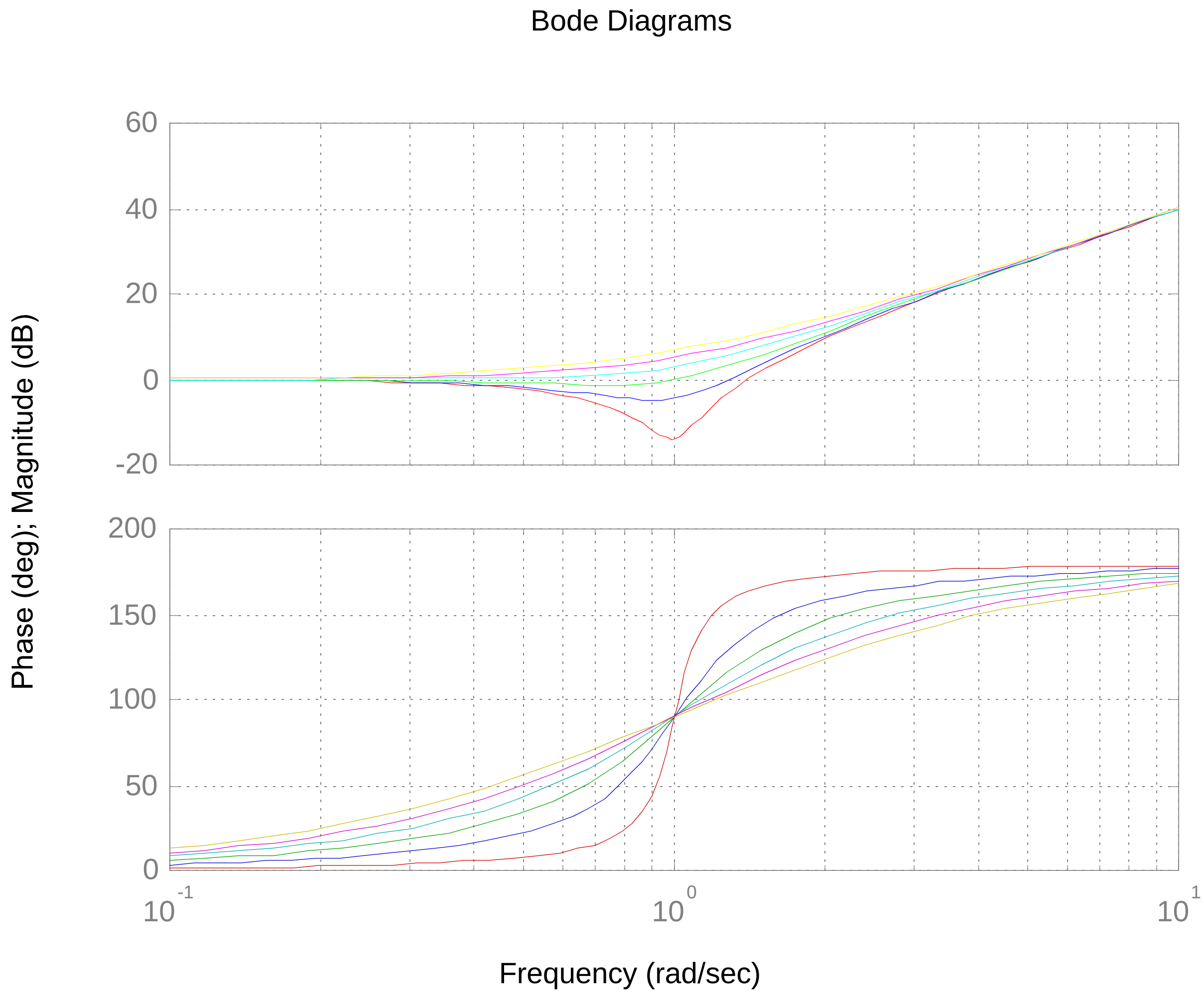
$$\text{for } u\ \ll\ 1 \quad \phi\ (j\omega)\ =\ 0°$$

$$\text{for } u\ \gg\ 1 \quad \phi\ (j\omega)\ =\ 180°$$

FREQUENCY RESPONSE ANALYSIS
Bode Plots

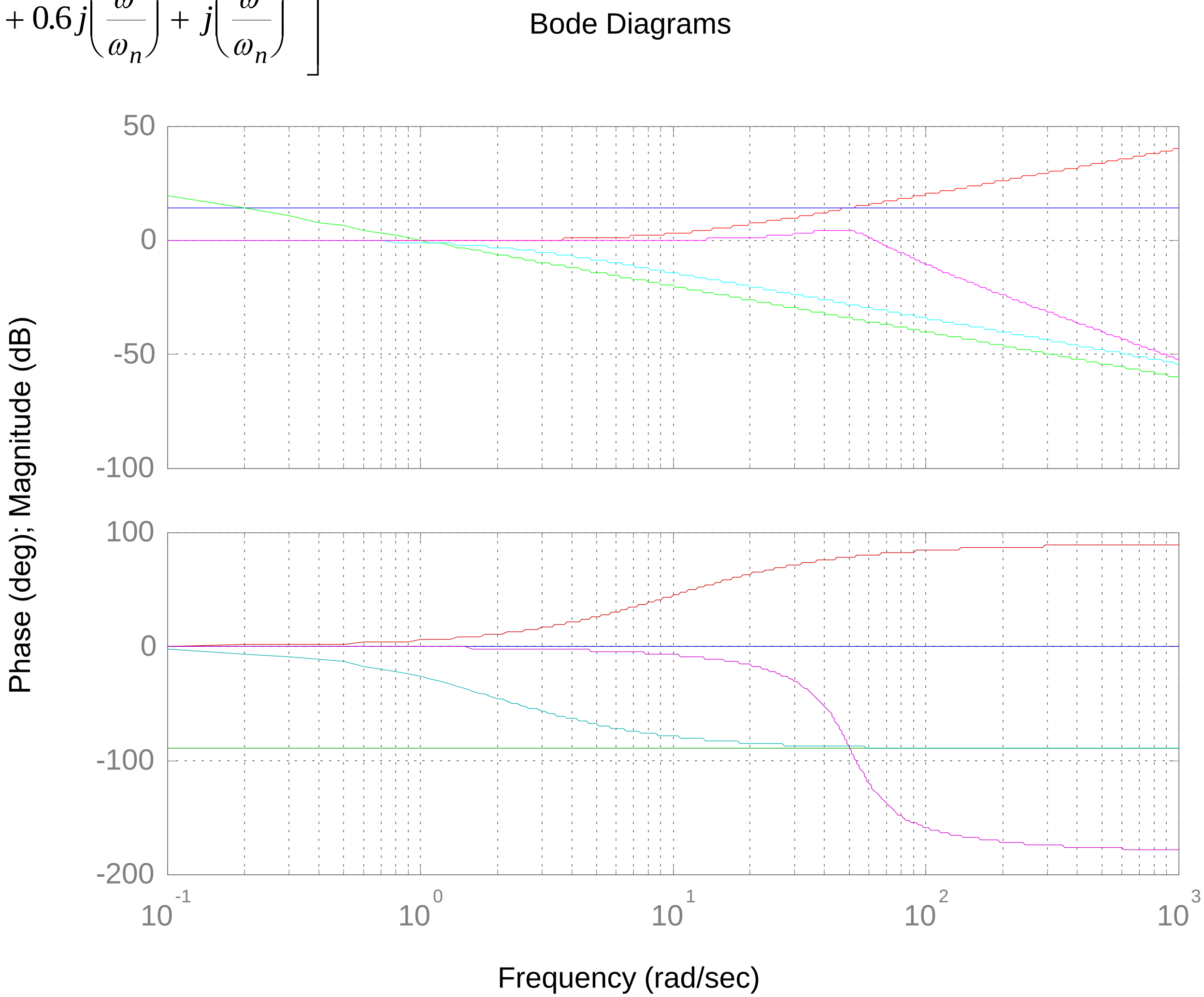
Complex Conjugate Zeros:

ζ = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1

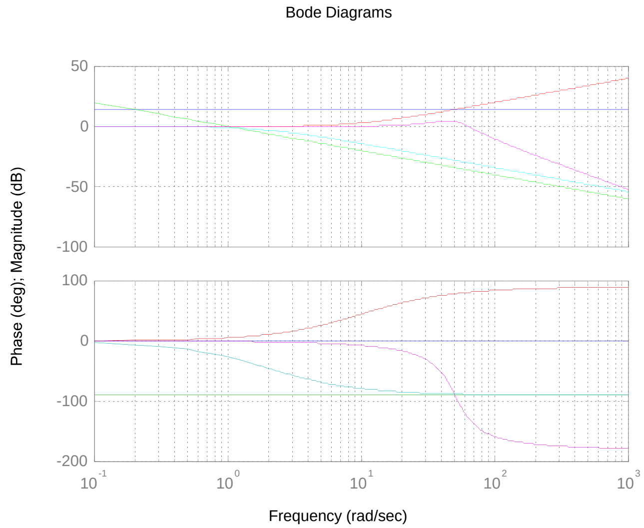


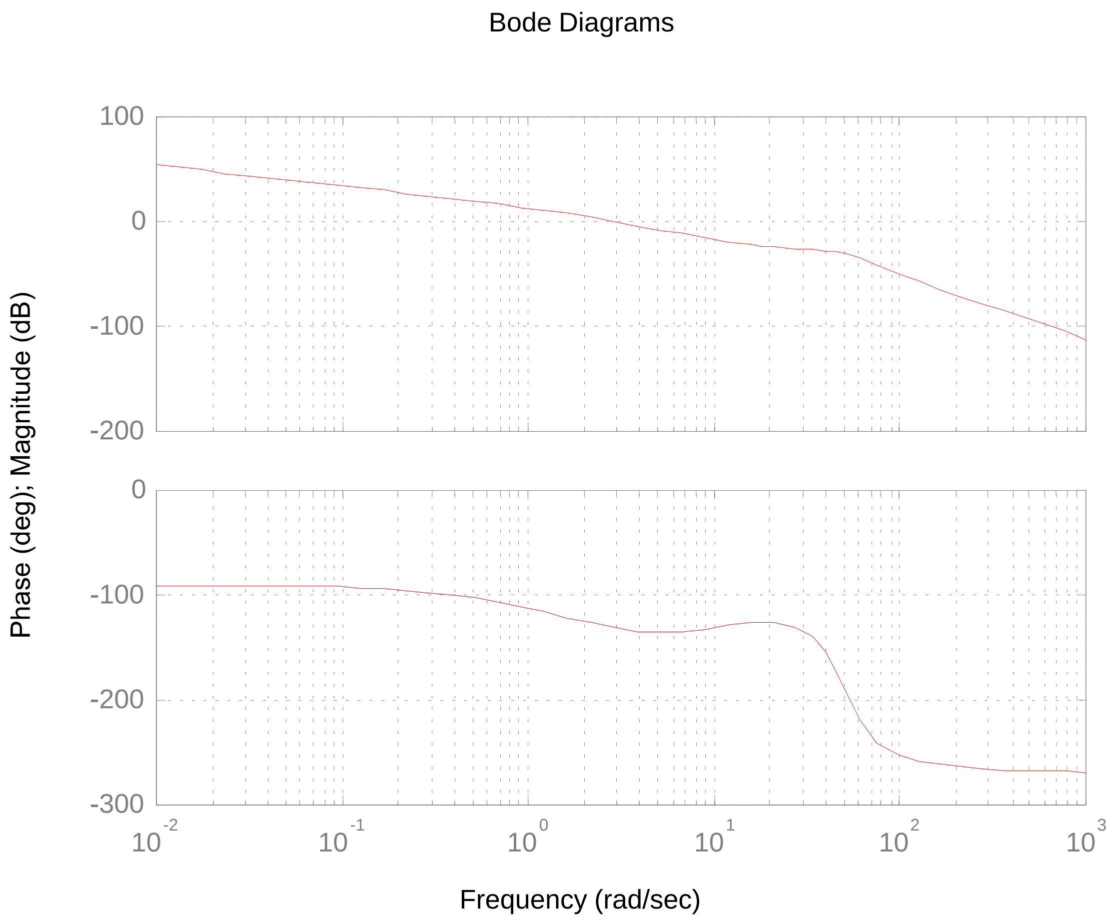
FREQUENCY RESPONSE ANALYSIS
Bode Plots

$$G(j\omega) = \frac{5[1+0.1j\omega]}{[j\omega][1+0.5j\omega]\left[1+0.6j\left(\frac{\omega}{\omega_n}\right)+j\left(\frac{\omega}{\omega_n}\right)^2\right]}$$



FREQUENCY RESPONSE ANALYSIS
Bode Plots





FREQUENCY RESPONSE ANALYSIS
Log-Magnitude Phase Plot

Log-magnitude phase plots contain the same information found in either the polar plots or Bode plots.

In this case, the log-magnitude in db is represented by the ordinate (y-axis) and the phase in degrees is represented by the abscissa. Frequency points are entered on the plot.

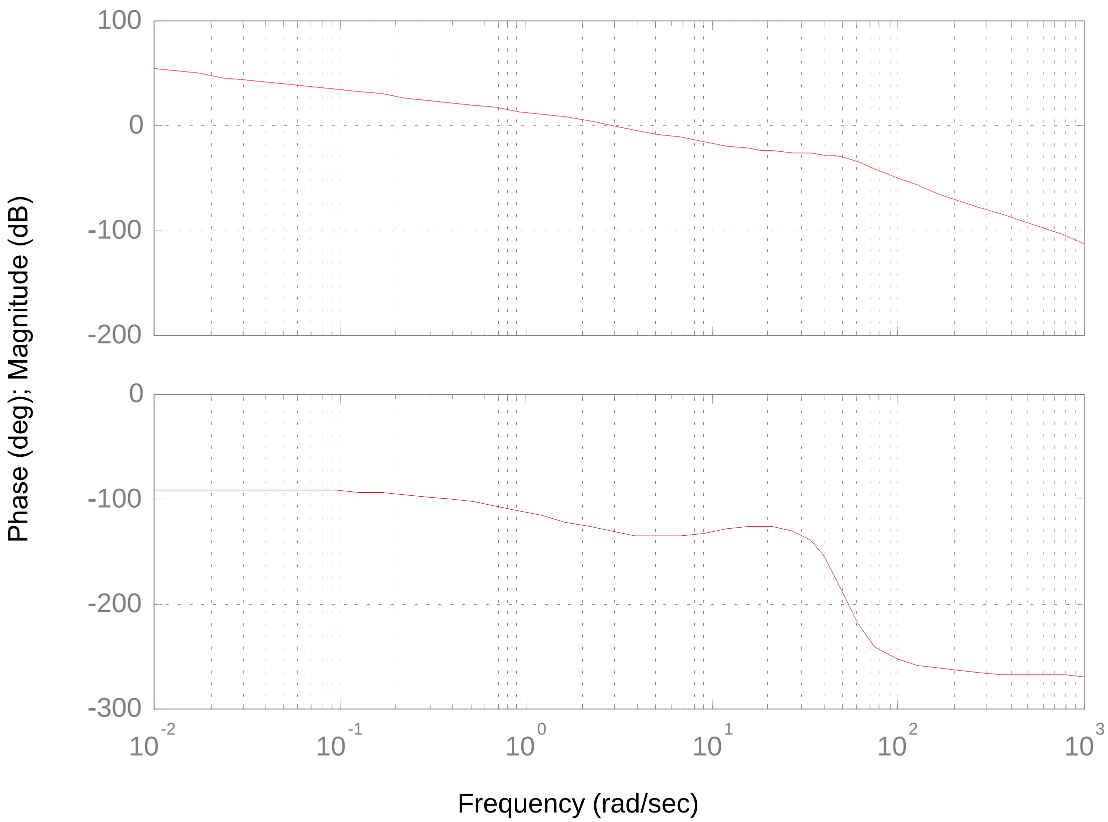
The log-magnitude phase plot is generally obtained from the Bode plot.

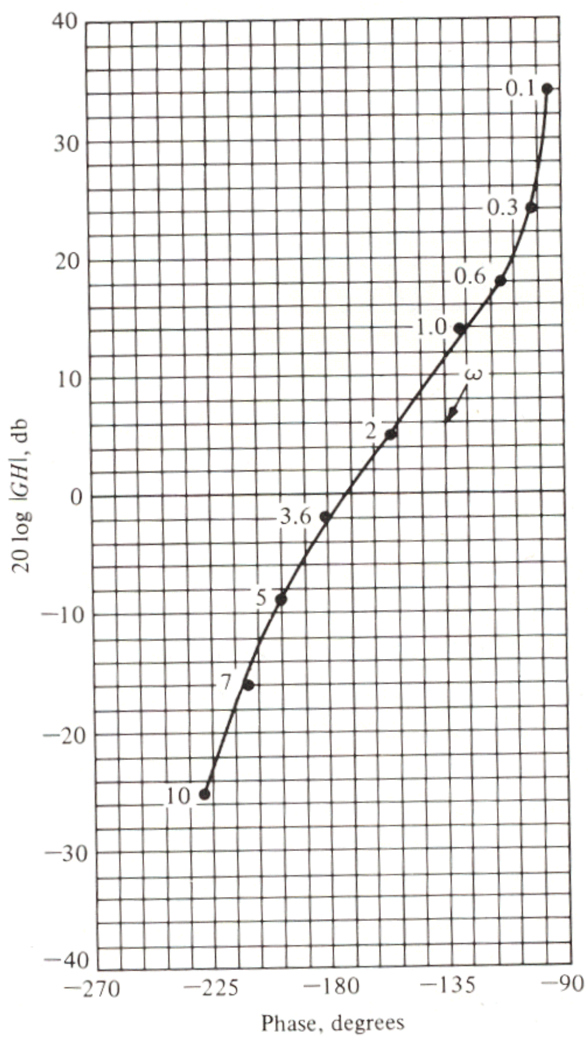
Any of the three plots (polar, Bode, or log-magnitude phase plot) can be obtained from any other.

FREQUENCY RESPONSE ANALYSIS
Log-Magnitude Phase Plot

$$G(j\omega) = \frac{5[1+0.1j\omega]}{[j\omega][1+0.5j\omega]\left[1+0.6j\left(\frac{\omega}{\omega_n}\right)+j\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Bode Diagrams





FREQUENCY RESPONSE ANALYSIS
Stability

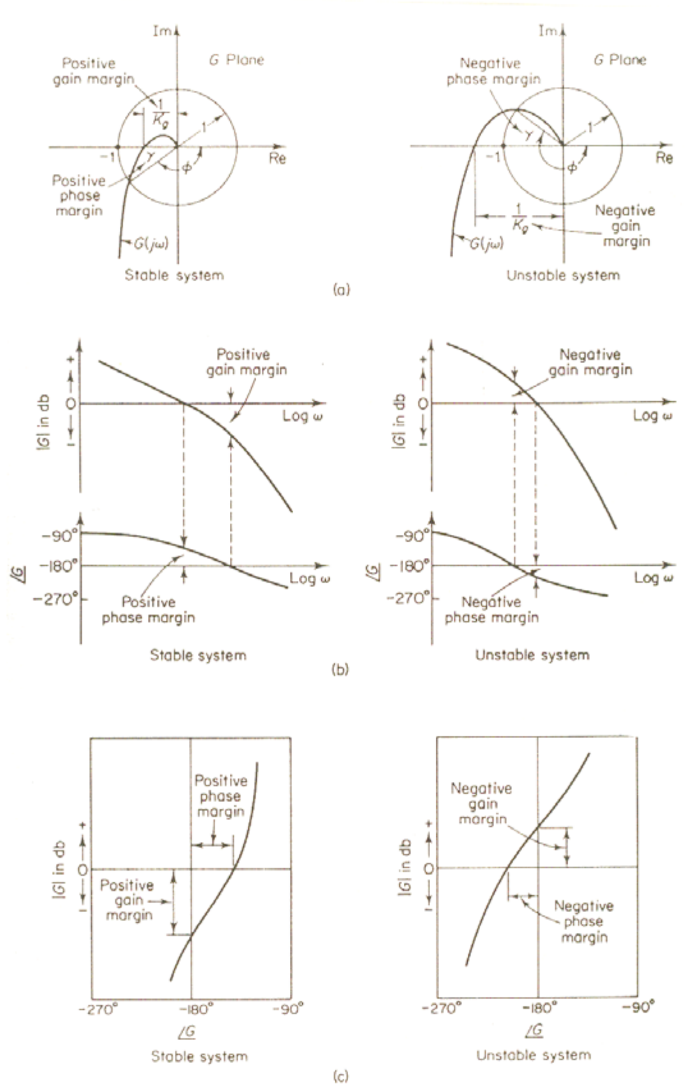
Gain Margin:

If the magnitude of the response is larger than that of the input -i.e. the gain is $+^{ve}$ db-when the phase shift is -180°, the system is unstable.

Phase Margin:

If the phase shift between the the response and the input is less -180° when response is equal to the input, i.e. the gain is 0 db, the system is unstable.

FREQUENCY RESPONSE ANALYSIS
Stability



Phase and gain margins of stable and unstable systems. (a) Polar plots; (b) logarithmic plots; (c) log-magnitude versus phase plots.