FREQUENCY RESPONSE ANALYSIS
Resonance

For the case of a set of complex conjugate poles and zeros, the magnitude plot near the natural frequency ω_n experience a maxima (for the case of a pole) or a minima (for the case of a zero) when the damping ratio ζ is less than about 0.7.

This peek does not occur exactly at ω_n but rather at a frequency slightly less than ω_n. This frequency is called the resonance frequency.

In order to obtain this resonance frequency we procede as follows:

FREQUENCY RESPONSE ANALYSIS
Resonance

Recall the magnitude calculation for a complex conjugate set of poles:

$$\text{Magnitude} \qquad 20\times\log\left|\frac{1}{1+2\zeta\, uj-u^2}\right| = 20\times\log\left|\frac{1}{(1-u^2)+2\zeta\, uj}\right|$$

$$=20\times\log\sqrt{\frac{(1-u^2)^2+(2\zeta\, u)^2}{\left[(1-u^2)^2+(2\zeta\, u)^2\right]^2}}$$

$$=20\times\log\sqrt{\frac{1}{(1-u^2)^2+(2\zeta\, u)^2}}$$

The magnitude expression reaches its maximum when the denominator under the radical reaches its minimum, we will call this denominator here *h*. Thus;

FREQUENCY RESPONSE ANALYSIS
Resonance

$$h = (1-u^2)^2 + (2\zeta\, u)^2$$
$$= 1 - 2u^2 + u^4 + 4\zeta^2 u^2$$

and

$$\frac{dh}{du} = -4u + 4u^3 + 8\zeta^2 u$$
$$= 0$$

Solving for *u* gives the minimum of the magnitude at;

$$u = \sqrt{1-2\zeta^2} \quad \text{or} \quad \omega_r = \omega_n \sqrt{1-2\zeta^2}$$

FREQUENCY RESPONSE ANALYSIS
Resonance

These last expressions are valid only between $\zeta = 0$ and $\zeta = 0.707$. Note that ζ cannot be less than zero, and $\zeta > 0.707$ yields an imaginary u_r.

The peak of the magnitude curve can be obtained by substituting for u_r in the magnitude expression, this gives:

$$\text{Magnitude} \qquad 20 \times \log \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$= -10 \times \log\left[4\zeta^2\left(1-\zeta^2\right)\right]$$

Note that at $\zeta = 0.707$ the peak magnitude is exactly 1, and at $\zeta = 0.0$ $\omega_r = \omega_n$ and the magnitude is infinte.

FREQUENCY RESPONSE ANALYSIS
Resonance

For the case of a set of complex conjugate zeros, the analysis is identical with the exception that instead of a peak the magnitude experiences a minimum.

Resonance still occur at ω_r as calculated before, and the magnitude will be the same but without the negative sign.

FREQUENCY RESPONSE ANALYSIS
Lead Compensator

Lead compensators were used in the case of the root locus analysis to modify the transient response of a system.

Lead compensators are also used here to alter the transient response of the a system.

It was shown in Chapter 7 that a lead network has a transfer function given by:

$R_1 C = \tau, \quad$ and $\dfrac{R_1}{R_1 + R_2} = \alpha$ (attenuation factor α<1), then

$$G(s) = \alpha \frac{\tau s + 1}{\alpha \tau s + 1}$$

$$= \frac{s + 1/\tau}{s + 1/\alpha\tau}$$

FREQUENCY RESPONSE ANALYSIS
Lead Compensator

In the frequency domain the lead network transfer function is given by:

$$G(j\omega) = \alpha \frac{(\omega\tau)\, j + 1}{(\omega\alpha\tau)\, j + 1}$$

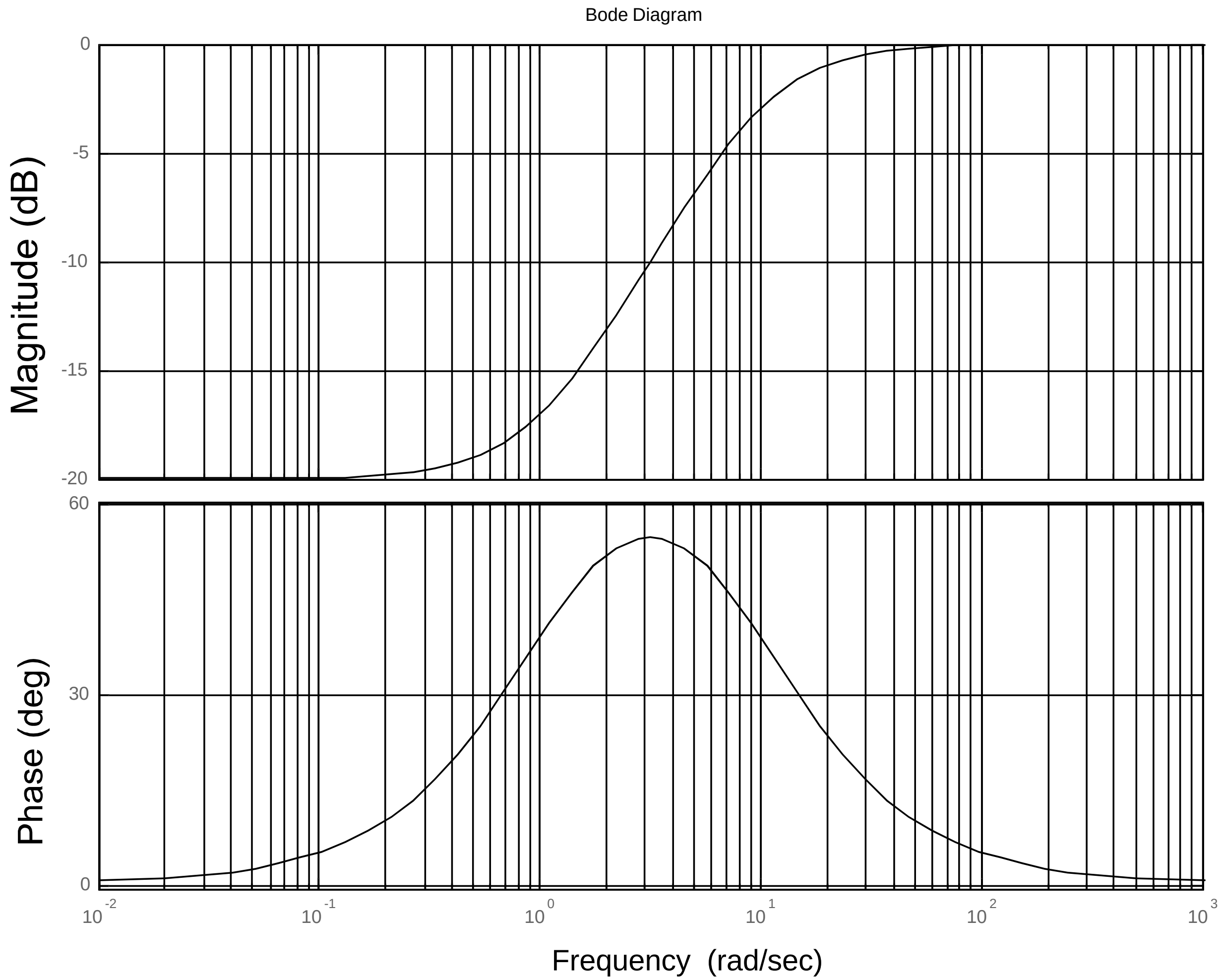
The pole and zero corner frequencies are given repectively by:

$$\omega_p = 1/(\alpha\tau) \quad \text{and} \quad \omega_z = 1/\tau$$

Since α is less than 1 in the case of a lead network, it follows that corner frequency of the pole is located to the right of the corner frequency of the zero on the Bode diagram, i.e.,

FREQUENCY RESPONSE ANALYSIS
Lead Compensator

$$0.1 \times \frac{j\omega + 1}{0.1\, j\omega + 1}$$



FREQUENCY RESPONSE ANALYSIS
Lead Compensator

The phase shift of the lead network is given by:

$$\phi(j\omega) = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$$

$$\therefore \tan\phi = \frac{\omega\tau - \alpha\omega\tau}{1+(\omega\tau)^2\alpha}$$

The frequency ω_m at which the maximum phase shift of a lead network occurs is obtained as follows:

$$\frac{d\phi}{d\omega} = 0 = \frac{1}{1+(\omega_m\tau)^2}\times\tau - \frac{1}{1+(\alpha\omega_m\tau)^2}\times\alpha\tau$$

$$\therefore \ \omega_m{}^2\alpha\tau^2(1-\alpha) = 1-\alpha$$

$$\therefore \ \omega_m = \frac{1}{\tau\sqrt{\alpha}}$$

FREQUENCY RESPONSE ANALYSIS
Lead Compensator

Since the corner frequency of the pole is $1/\alpha$ away from the corner frequency of the zero, and ω_m is at $1/\alpha^{0.5}$, it follows that ω_m is the geometric mean of ω_p and ω_z. On the logarethmic scale ω_m is at the mid point between ω_p and ω_z.

Substituting ω_m into ϕ equation yields:

$$\tan\phi_m = \frac{1/\sqrt{\alpha} - \sqrt{\alpha}}{1+1}$$

$$= \frac{1-\alpha}{2\sqrt{\alpha}}$$

$$\therefore \ \sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

FREQUENCY RESPONSE ANALYSIS
Lead Compensator

At the maximum phase change of the lead network, the magnitude can be obtained by substituting ω_m in the magnitude equation as follows:

$$\left|\frac{1+j\omega\tau}{1+j\alpha\omega\tau}\right|_{\omega_m=\frac{1}{\tau\sqrt{\alpha}}} = \frac{\left(1+j\,{}^{1}\!/\!\sqrt{\alpha}\right)\left(1-j\sqrt{\alpha}\right)}{\left(1+j\sqrt{\alpha}\right)\left(1-j\sqrt{\alpha}\right)}$$

$$= \frac{1+j\left(1-\alpha\right)/\sqrt{\alpha}+1}{1+\alpha}$$

$$= \frac{1}{\sqrt{\alpha}}$$

FREQUENCY RESPONSE ANALYSIS
Lag Compensator

Lag compensators were used in the case of the root locus analysis to modify the steady state response of a system by increasing the effectively gain of the system without altering the transient response as increasing the system gain would do.

In the frequency domain lag compensators are designed and used to alter the transient response of the a system.

It was shown in Chapter 7 that a lag network has a transfer function given by:

$$R_2 C = \tau, \quad \text{and} \quad \frac{R_1 + R_2}{R_2} = \beta \text{ (amplification factor } \beta > 1\text{), then}$$

$$G(s) = \frac{\tau s + 1}{\beta \tau s + 1}$$

$$= \frac{1}{\beta} \frac{s + 1/\tau}{s + 1/\beta\tau}$$

FREQUENCY RESPONSE ANALYSIS
Lag Compensator

In the frequency domain the lag network transfer function is given by:

$$G(j\omega) = \frac{1}{\beta}\frac{(\omega\tau)\,j + 1}{(\omega\beta\tau)\,j + 1}$$

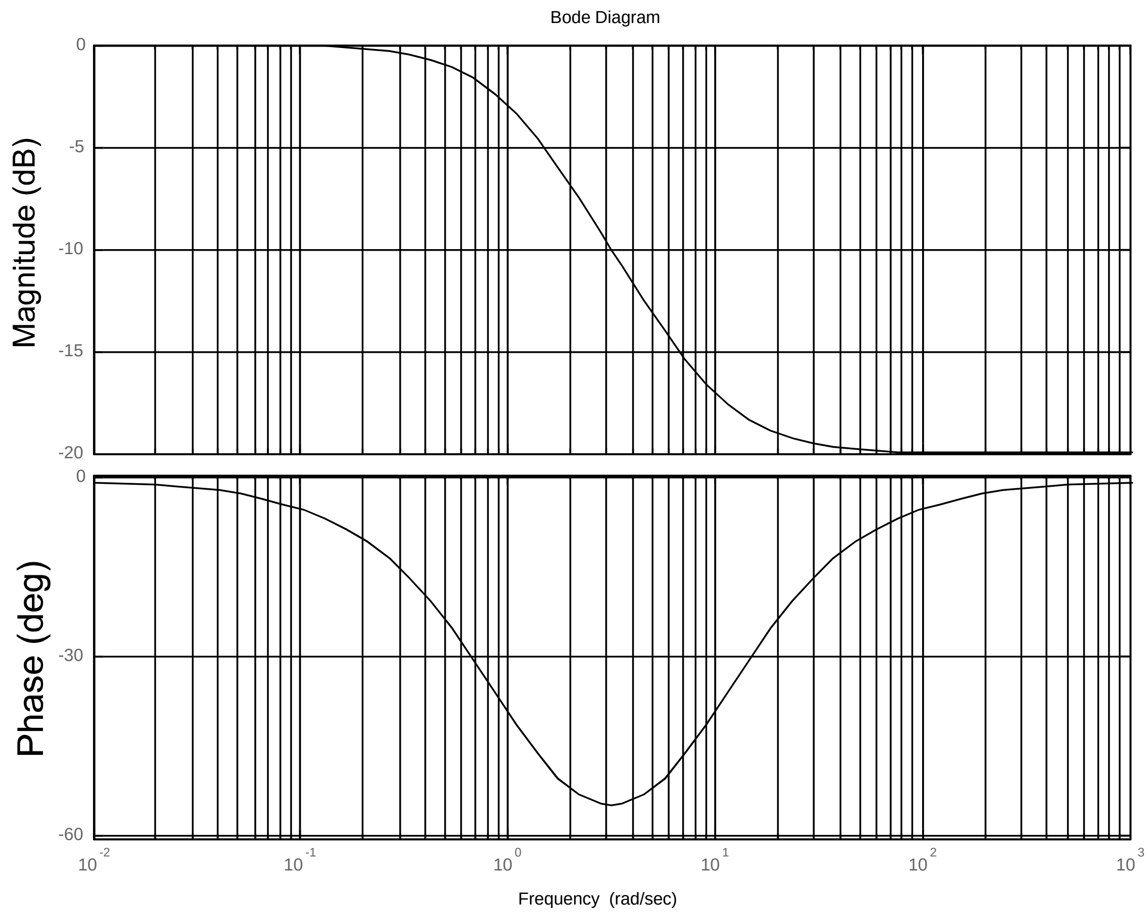
The pole and zero corner frequencies are given repectively by:

$$\omega_p = 1/(\beta\tau) \quad \text{and} \quad \omega_z = 1/\tau$$

Since β is greater than 1 in the case of a lag network, it follows that corner frequency of the pole is located to the left of the corner frequency of the zero on the Bode diagram, i.e.,

FREQUENCY RESPONSE ANALYSIS
Lag Compensator

$$\frac{j0.1\omega + 1}{j\omega + 1}$$



FREQUENCY RESPONSE ANALYSIS
Lag Compensator

The phase shift of the lag network is given by:

$$\phi(j\omega) = \tan^{-1}(\omega\tau) - \tan^{-1}(\beta\omega\tau)$$

$$\therefore \tan\phi = \frac{\omega\tau - \beta\omega\tau}{1+(\omega\tau)^2\beta}$$

The frequency ω_m at which the maximum phase shift of a lag network occurs is obtained as follows:

$$\frac{d\phi}{d\omega} = 0 = \frac{1}{1+(\omega_m\tau)^2}\times\tau - \frac{1}{1+(\beta\omega_m\tau)^2}\times\beta\tau$$

$$\therefore\ \omega_m{}^2\beta\tau^2(1-\beta) = 1-\beta$$

$$\therefore\ \omega_m = \frac{1}{\tau\sqrt{\beta}}$$

FREQUENCY RESPONSE ANALYSIS
Lag Compensator

Since the corner frequency of the pole is $1/\alpha$ away from the corner frequency of the zero, and ω_m is at $1/\beta^{0.5}$, it follows that ω_m is the geometric mean of ω_p and ω_z. On the logarethmic scale ω_m is at the mid point between ω_p and ω_z.

Substituting ω_m into ϕ equation yields:

$$\tan\phi_m = \frac{1/\sqrt{\beta} - \sqrt{b}}{1+1}$$

$$= \frac{1-b}{2\sqrt{\beta}}$$

$$\therefore \ \sin\phi_m = \frac{1-\beta}{1+\beta}$$